

Complex ADHM equations,
sheaves on \mathbb{P}^3
and quantum instantons

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Abstract

We use a complex version of the celebrated Atiyah-Hitchin-Drinfeld-Manin matrix equations to construct admissible torsion-free sheaves on \mathbb{P}^3 and complex quantum instantons over our quantum Minkowski space-time. We identify the moduli spaces of various subclasses of sheaves on \mathbb{P}^3 , and prove their smoothness. We also define the Laplace equation in the quantum Minkowski space-time, study its solutions and relate them to the admissibility condition for sheaves on \mathbb{P}^3 .

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Introduction

In [4] we introduced a quantum Minkowski space-time based on the quantum group $SU(2)_q$ extended by a degree operator, and we formulated a quantum version of the anti-self-dual Yang-Mills (ASDYM) equation with unitary gauge group. A remarkable feature of the quantum equations is the natural parameterization of their solutions of fixed rank r and charge c by the classical Atiyah-Hitchin-Drinfeld-Manin (ADHM) moduli spaces $\mathcal{M}^{\text{reg}}(r, c)$.

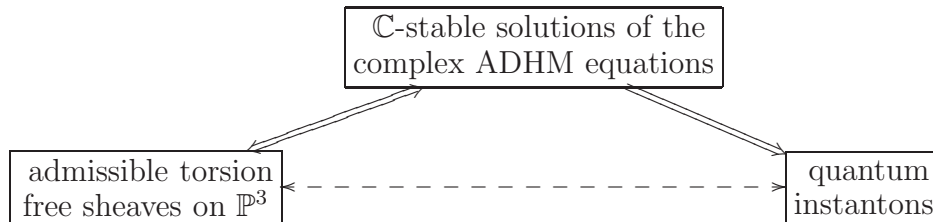
The theory of ASDYM equations substantially involves the complex structures, including the fundamental twistor space of Penrose. It is therefore appropriate to study also the ASDYM equations with a complex linear gauge group on the complexified Minkowski space-time (see e.g. [16]). The corresponding counterparts of the ADHM moduli spaces $\mathcal{M}^{\text{reg}}(r, c)$ were implicitly described by Donaldson in [3], who in fact used them to construct certain holomorphic vector bundles on \mathbb{P}^3 related to instantons by the Penrose twistor diagram. These holomorphic vector bundles admit a cohomological characterization [8], and are known in the literature as complex instanton bundles [12].

In this paper, we expand the classical theory of the complex ASDYM equations in two related directions. On one hand, we will consider the moduli spaces of complex *torsion free instanton sheaves* on \mathbb{P}^3 and show that they correspond to extending the moduli spaces of complex instantons $\mathcal{M}_{\mathbb{C}}^{\text{reg}}(r, c)$ to the larger moduli spaces of stable complex ADHM data, denoted by $\mathcal{M}_{\mathbb{C}}(r, c)$. On the other hand, we will use the stable complex ADHM data to construct *complex quantum instantons* over the complexified quantum Minkowski space-time introduced in [4]. In this paper we also generalize the classical relation [8] between the admissibility condition that determines the instanton bundles and the solutions of the Laplace equations by developing the quantum differential calculus.

The organization of the paper is completely reflected in the title, and consists of three sections complemented by three appendices. In Section 1

we define the complex ADHM equations and the associated moduli spaces. We prove various results about them, including smoothness, dimension and non-emptiness of $\mathcal{M}_{\mathbb{C}}(r, c)$ and its open subset $\mathcal{M}_{\mathbb{C}}^{\text{reg}}(r, c)$ for $r \geq 2$. In Section 2, we define and study admissible torsion-free sheaves on \mathbb{P}^3 . The main result of this section is the characterization of the sheaves corresponding to $\mathcal{M}_{\mathbb{C}}(r, c)$. Using such characterization we show that the spaces $\mathcal{M}_{\mathbb{C}}(1, c)$ are empty. We also identify the open subsets of $\mathcal{M}_{\mathbb{C}}(r, c)$ that characterize reflexive and locally-free admissible sheaves. In particular, we verify that the moduli spaces of framed complex instanton bundles of rank r and charge c over \mathbb{P}^3 is given by the regular complex ADHM data $\mathcal{M}_{\mathbb{C}}^{\text{reg}}(r, c)$ and has the structure of a smooth complex manifold of dimension $4rc$. In Section 3 we recall the definition of the quantum Minkowski space-time and the quantum ASDYM equations, and we construct quantum instantons from the ADHM data parameterized by the moduli spaces $\mathcal{M}_{\mathbb{C}}(r, c)$. At the last subsection we define the quantum Laplace equation and produce their solutions from the cohomology of an instanton sheaf on \mathbb{P}^3 . This extends the phenomenon observed in our first paper [4], in the case of anti-self-dual Yang-Mills equation: the solutions of the *quantum equation* are parameterized by the *classical data*. We thus obtain a surprisingly *explicit and direct link between the commutative geometry encoded into sheaves on \mathbb{P}^3 and the noncommutative geometry of the quantum Minkowski space-time*. We also discuss the consistency of the Laplace equations on the two affine parts $\mathfrak{M}_q^{\text{I}}$ and $\mathfrak{M}_q^{\text{J}}$ of our quantum Minkowski space-time, thus relating the admissibility condition with the non-existence of consistent solutions to the quantum Laplace equation. Finally, in the three appendices corresponding to each of the three sections we collect various background facts and calculations used throughout the paper.

Our constructions are best reflected by the following triangle, whose vertices correspond to the objects studied in the three sections:



The solid arrows were established in this paper; the dashed arrow corresponds to the quantum Penrose transform introduced in [4] and which we plan to study further in a sequel to this paper. The completion of this circle of ideas will yield a complete characterization of quantum instantons, while opening, at the same time, a new perspective in the theory of sheaves on \mathbb{P}^3 . It should also enhance the direct connection of the commutative geometry encoded into sheaves on \mathbb{P}^3 and the noncommutative geometry of the quantum Minkowski space-time.

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1 Complex ADHM equations

We start with some fundamental definitions and further motivation, before formulating our first main result in Section 1.2.

1.1 ADHM data and instantons

Let V and W be complex vector spaces, with dimensions c and r , respectively, and consider maps $B_1, B_2 \in \text{End}(V)$, $i \in \text{Hom}(W, V)$ and $j \in \text{Hom}(V, W)$. This so-called *ADHM datum* (B_1, B_2, i, j) is said to be:

1. *stable*, if there is no proper subspace $S \subset V$ such that $B_k(S) \subset S$ ($k = 1, 2$) and $i(W) \subset S$;

2. *costable*, if there is no proper subspace $S \subset V$ such that $B_k(S) \subset S$ ($k = 1, 2$) and $S \subset \ker j$;
3. *regular*, if it is both stable and costable.

Notice that (B_1, B_2, i) is stable if and only if the triple (B_1^*, B_2^*, i^*) consisting of the dual maps is costable.

Now, providing V and W with Hermitian structures, we consider the so-called *ADHM equations* (\dagger denotes Hermitian conjugation):

$$[B_1, B_2] + ij = 0 \quad (1)$$

$$[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + ii^\dagger - j^\dagger j = \xi \mathbf{1}_V \quad (2)$$

With $GL(V)$ acting on the set of all ADHM data in the following way:

$$g(B_{kl}, i_k, j_k) = (gB_{kl}g^{-1}, gi_k, j_lg^{-1}), \quad g \in GL(V)$$

we define the following varieties:

$$\mathcal{M}^0(r, c) = \{\text{all solutions of (1) and (2) with } \xi = 0\} / U(V)$$

$$\begin{aligned} \mathcal{M}(r, c) &= \{\text{stable solutions of (1)}\} / GL(V) \simeq \\ &\simeq \{\text{all solutions of (1) and (2) with } \xi > 0\} / U(V) \end{aligned}$$

$$\begin{aligned} \mathcal{M}^{\text{reg}}(r, c) &= \{\text{regular solutions of (1)}\} / GL(V) \simeq \\ &\simeq \{\text{regular solutions of (1) and (2) with } \xi = 0\} / U(V) \end{aligned}$$

It can be shown that $\mathcal{M}^{\text{reg}}(r, c)$ and $\mathcal{M}(r, c)$ are smooth hyperkähler manifolds of dimension $2rc$, with the exception that $\mathcal{M}^{\text{reg}}(r, c)$ is empty for $r = 1$; the proofs are given in Appendix A for they are used in the next section. Moreover, $\mathcal{M}^{\text{reg}}(r, c)$ is the smooth locus of the singular variety $\mathcal{M}^0(r, c)$ for $r \geq 2$, while $\mathcal{M}(r, c)$ is the minimal resolution of $\mathcal{M}^0(r, c)$ [9, 10].

1.2 Complex ADHM data

As above, let V and W be complex vector spaces, with dimensions c and r , respectively; Hermitian structures are no longer required. Set $\tilde{W} = V \oplus V \oplus W$, and define also:

$$\mathbf{B} = \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

Let $\mathbb{B} = \mathbf{B} \oplus \mathbf{B}$, with $\vec{B} = (B_{kl}, i_k, j_k) \in \mathbb{B}$ ($k, l = 1, 2$) being called a *complex ADHM datum*. As usual, the group $GL(V)$ acts naturally on \mathbf{B} and on \mathbb{B} , in the following way:

$$g(B_{kl}, i_k, j_k) = (gB_{kl}g^{-1}, gi_k, j_kg^{-1}), \quad g \in GL(V) \quad (3)$$

Equivalently, we can think of an element in \mathbb{B} as a holomorphic section of the bundle $\mathbf{B} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ by defining:

$$\tilde{B}_1 = zB_{11} + wB_{21} \quad \text{and} \quad \tilde{B}_2 = zB_{12} + wB_{22} \quad (4)$$

$$\tilde{i} = zi_1 + wi_2 \quad \text{and} \quad \tilde{j} = zj_1 + wj_2 \quad (5)$$

In other words, $\mathbb{B} = \mathbf{B} \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, with z, w denoting a basis of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ or, equivalently, a choice of homogeneous coordinates in \mathbb{P}^1 . In particular, one can also view the maps (4) and (5) in the following way:

$$\tilde{B}_1, \tilde{B}_2 \in \text{Hom}(V, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

$$\tilde{i} \in \text{Hom}(W, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \quad \text{and} \quad \tilde{j} \in \text{Hom}(V, W) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

The evaluation map $\text{ev}_p : H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \mathbb{C}$ can be tensored with the identity to yield maps $\text{ev}_p : \mathbb{B} \rightarrow \mathbf{B}$ and $\text{ev}_p : \text{Hom}(V, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \rightarrow \text{Hom}(V, V)$. For simplicity, we use the notation $\vec{B}_p = \text{ev}_p(\vec{B})$.

Definition. A complex ADHM datum $\vec{B} = (B_{kl}, i_k, j_k)$ is said to be:

1. \mathbb{C} -semistable if there is $p \in \mathbb{P}^1$ such that \vec{B}_p is stable;

2. \mathbb{C} -stable if \vec{B}_p is stable for all $p \in \mathbb{P}^1$;
3. \mathbb{C} -costable if \vec{B}_p is costable for all $p \in \mathbb{P}^1$;
4. \mathbb{C} -semiregular if it is \mathbb{C} -stable and there is $p \in \mathbb{P}^1$ such that \vec{B}_p is regular;
5. \mathbb{C} -regular if \vec{B}_p is regular for all $p \in \mathbb{P}^1$.

The motivation behind these definitions will be clearer in the next Section: \mathbb{C} -stable, \mathbb{C} -semiregular, and \mathbb{C} -regular will correspond to torsion-free, reflexive and locally-free sheaves on \mathbb{P}^3 , respectively. In particular, notice that $\vec{B} = (B_{kl}, i_k, j_k)$ is \mathbb{C} -regular if and only if it is both \mathbb{C} -stable and \mathbb{C} -costable. Moreover, if $\vec{B} = (B_{kl}, i_k, j_k)$ is \mathbb{C} -stable then $(B_{11}, B_{12}, i_1, j_1)$ and $(B_{21}, B_{22}, i_2, j_2)$ are both stable.

Proposition 1. *If \vec{B} is \mathbb{C} -semistable, then its $GL(V)$ stabilizer is trivial.*

Proof. If \vec{B} is fixed by some nontrivial $g \in GL(V)$, then \vec{B}_p is also fixed by g for all $p \in \mathbb{P}^1$. Thus, by Proposition 34 in Appendix A, there is no $p \in \mathbb{P}^1$ such that \vec{B}_p is stable. \square

The first main goal of this paper is to study the *complex ADHM equations*:

$$[B_{11}, B_{12}] + i_1 j_1 = 0 \quad (6)$$

$$[B_{21}, B_{22}] + i_2 j_2 = 0 \quad (7)$$

$$[B_{11}, B_{22}] + [B_{21}, B_{12}] + i_1 j_2 + i_2 j_1 = 0 \quad (8)$$

which were first posed by Donaldson in [3]; it is important to note that the equations (6-8) are equivalent to:

$$[\tilde{B}_1, \tilde{B}_2] + \tilde{i}\tilde{j} = 0, \quad \forall [z : w] \in \mathbb{P}^1 \quad (9)$$

It is easy to see that solutions of (6-8) are preserved by the $GL(V)$ action (3). Therefore, we define the moduli space of solutions of the complex ADHM

equations as the quotient:

$$\mathcal{M}_{\mathbb{C}}(r, c) := \left\{ \begin{array}{c} \mathbb{C} - \text{stable} \\ \text{solutions of (6-8)} \end{array} \right\} / GL(V)$$

Our first main result, to be proved in this Section, states that $\mathcal{M}_{\mathbb{C}}(r, c)$ is a smooth, complex manifold of complex dimension $4rc$, non-empty for $r \geq 2$. The strategy of the proof is the same as for Theorem 33 in Appendix A ; we consider the map

$$\begin{aligned} \tilde{\mu} : \mathbb{B}^{\text{st}} &\rightarrow \text{Hom}(V, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \\ \tilde{\mu}(B_{kl}, i_k, j_k) &= [\tilde{B}_1, \tilde{B}_2] + \tilde{i}\tilde{j} \end{aligned}$$

where \mathbb{B}^{st} denote the open subset of \mathbb{C} -stable complex ADHM data. Clearly $\mathcal{M}_{\mathbb{C}}(r, c) = \tilde{\mu}^{-1}(0)/GL(V)$. We have already established that $GL(V)$ acts freely on \mathbb{B}^{st} ; we must then argue that $\tilde{\mu}$ has a surjective derivative and that the $GL(V)$ action has a closed graph.

Proposition 2. *\vec{B} is \mathbb{C} -stable if and only if the derivative map*

$$D_{\vec{B}}\tilde{\mu} : \mathbb{B} \rightarrow \text{Hom}(V, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2))$$

is surjective.

Proof. Considering the maps ($k = 1, 2$):

$$\begin{aligned} \partial_k : \mathbf{B} &\rightarrow \text{Hom}(V, V) \\ \partial_k &= [\cdot, B_{k2}] + [B_{k1}, \cdot] + i_k \cdot + \cdot j_k \end{aligned}$$

which can also be regarded as $c^2 \times (2c^2 + 2cr)$ matrix. We can then express the $3c^2 \times (4c^2 + 4cr)$ matrix of the derivative map $D_{\vec{B}}\tilde{\mu}$ in the following form:

$$D_{\vec{B}}\tilde{\mu} = \begin{pmatrix} \partial_1 & 0 \\ \partial_2 & \partial_1 \\ 0 & \partial_2 \end{pmatrix} \tag{10}$$

Our goal is to show that the above matrix has maximal rank $3c^2$ if and only if \vec{B} is \mathbb{C} -stable.

So let $\{l_k\}_{k=1}^{c^2}$ denote the rows of the matrix ∂_1 , and let $\{l'_k\}_{k=1}^{c^2}$ denote the rows of the matrix ∂_2 ; each l_k, l'_k is regarded as a vector in \mathbf{B} . As remarked above, \mathbb{C} -stability is equivalent to the vectors $\{zl_k + wl'_k\}_{k=1}^{c^2}$ being linearly independent (as vectors in \mathbf{B}) for all $[z : w] \in \mathbb{P}^1$. The rows of the matrix $(\partial_2 \ \partial_1)$ are then given by $\{(l'_k, l_k)\}_{k=1}^{c^2}$ regarded as vectors in $\mathbf{B} \oplus \mathbf{B}$. Clearly, the matrix (10) above is surjective if and only if $\{(l_k, 0), (l'_k, l_k), (0, l'_k)\}_{k=1}^{c^2}$ form a linearly independent set of vectors in \mathbb{B} ; this is in turns equivalent to the statement that if the coefficients γ^k are such that $\sum_k \gamma^k l_k \in \text{span}\{l'_k\}$ and $\sum_k \gamma^k l'_k \in \text{span}\{l_k\}$, then $\gamma^k = 0$.

Let $L = \text{span}\{l_k\}_{k=1}^{c^2}$, $L' = \text{span}\{l'_k\}_{k=1}^{c^2}$; the theorem can then be reduced to the following statement: the vectors $\{zl_k + wl'_k\}_{k=1}^{c^2}$ are linearly independent for all $[z : w] \in \mathbb{P}^1$ if and only if $\sum_k \gamma^k l_k \in L'$ and $\sum_k \gamma^k l'_k \in L$ implies $\gamma^k = 0$.

First, assume that if $\sum_k \gamma^k l_k \in L'$ and $\sum_k \gamma^k l'_k \in L$, then $\gamma^k = 0$. If $\sum_k \gamma^k (zl_k + wl'_k) = 0$, then $L \ni \sum_k \gamma^k l_k = -\frac{w}{z} \sum_k \gamma^k l'_k \in L'$; hence $\gamma^k = 0$, and $\{zl_k + wl'_k\}_{k=1}^{c^2}$ is linearly independent for all $[z : w] \in \mathbb{P}^1$.

For the converse direction, denote $I = L \cap L'$; we can assume that $I = \text{span}\{l_k\}_{k=1}^d = \text{span}\{l'_k\}_{k=1}^d$. Since $\sum_k \gamma^k l_k, \sum_k \gamma^k l'_k \in I$, we have $\gamma^k = 0$ for $k = d+1, \dots, c^2$. Moreover, for each $k = 1, \dots, d$, we have $l'_k = \sum_{n=1}^d g_k^n l_n$; let G be the invertible $d \times d$ matrix with entries g_k^n . Therefore,

$$\sum_{k=1}^d \gamma^k (zl_k + wl'_k) = \sum_{k,n=1}^d \gamma^k (z\delta_k^n + wg_k^n) l_n = \sum_{n=1}^d c^n(z, w) l_n,$$

where $c^n(z, w)$ are the entries of the vector $\gamma(z\mathbf{1} + wG)$; note that, by construction, the matrix $(z\mathbf{1} + wG)$ is also invertible for generic $[z : w] \in \mathbb{P}^1$. Now, the vectors $\{l_n\}$ are linearly independent, so if $\sum_{k=1}^d \gamma^k (zl_k + wl'_k) = 0$, then $c_n(z, w) = 0$ for each $n = 1, \dots, d$ and for all $[z : w] \in \mathbb{P}^1$. This implies that $\gamma^k = 0$, as desired. \square

As a by-product of our proof, we obtain the following interesting result:

Proposition 3. *If the complex ADHM datum \vec{B} is:*

- \mathbb{C} -semistable, then there are at most finitely many points $p \in \mathbb{P}^1$ such that \vec{B}_p is not stable.
- \mathbb{C} -semiregular, then there are at most finitely many points $p \in \mathbb{P}^1$ such that \vec{B}_p is not regular.

Proof. Assume that \vec{B}_p is stable for some $p = [p_1 : p_2] \in \mathbb{P}^1$. Then $p_1\partial_1 + p_2\partial_2$ is surjective and its row vectors $\{p_1l_k + p_2l'_k\}_{k=1}^{c^2}$ are linearly independent as vectors in \mathbf{B} . We complete $\{p_1l_k + p_2l'_k\}_{k=1}^{c^2}$ to a basis of \mathbf{B} , and denote by $H(p)$ the $(2c^2 + 2cr) \times (2c^2 + 2cr)$ matrix formed by such basis.

As we vary $p \in \mathbb{P}^1$, we see that $\det H(p)$ is a section of $\mathcal{O}_{\mathbb{P}^1}(c^2)$. Hence $\{zl_k + wl'_k\}_{k=1}^{c^2}$ must be linearly independent except for finitely many $[z : w] \in \mathbb{P}^1$, which means that $\vec{B}_{[z:w]}$ is stable away from finitely many (up to c^2) points in \mathbb{P}^1 .

The second statement follows by duality. \square

Proposition 4. *The action (3) has a closed graph.*

Proof. Let $\{X_k\}$ be a sequence in \mathbb{B}^{st} , while $\{g_k\}$ denotes a sequence in $GL(V)$; assuming that

$$\lim_{k \rightarrow \infty} X_k = X_\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} g_k \cdot X_k = Y_\infty ,$$

we show that the sequence $\{g_k\}$ converges to some $g_\infty \in GL(V)$, so that $Y_\infty = g_\infty \cdot X_\infty$.

Using evaluation at $p \in \mathbb{P}^1$, it follows that:

$$\lim_{k \rightarrow \infty} (X_k)_p = (X_\infty)_p \quad \text{and} \quad \lim_{k \rightarrow \infty} g_k \cdot (X_k)_p = (Y_\infty)_p .$$

Hence, by argument in the proof of Proposition 35 in Appendix A, we conclude that

$$\lim_{k \rightarrow \infty} g_k = g_\infty = R((Y_\infty)_p) (T_\infty)_p [R((X_\infty)_p) (T_\infty)_p]^{-1} \in GL(V) ,$$

where the map $R(X) : W^{\oplus c^2} \longrightarrow V$ given, for $X = (B_1, B_2, i, j) \in \mathbf{B}^{\text{st}}$, by:

$$R(X) = i \oplus \cdots \oplus B_1^m B_2^n i \oplus \cdots \oplus B_1^{c-1} B_2^{c-1} i \quad , \quad 1 \leq m, n \leq c-1 \quad .$$

Note that $gR(X) = R(g \cdot X)$ for any $g \in GL(V)$, and that $R(X)$ is surjective if and only if X is stable. \square

1.3 Existence of solutions

So far, we can conclude from Propositions 1 and 2 that $\mathcal{M}_{\mathbb{C}}(r, c)$ is a smooth complex manifold of dimension $4rc$ provided it is non-empty. The case $r = \dim W = 1$ is rather special due to the following result:

Proposition 5. *There are no \mathbb{C} -stable solutions of (6-8) for $r = 1$.*

The proof will be delayed until the end of Section 2.1 (see Proposition 14). However, let us consider here the simplest possible case: $r = c = 1$. In this case, a \mathbb{C} -stable solution of (6-8) reduces to six complex numbers (b_{kl}, i_k) , since $j_1 = j_2 = 0$ by Proposition 38. Now \tilde{i} is simply a section of $\mathcal{O}_{\mathbb{P}^1}(1)$, so it must vanish at one point $p \in \mathbb{P}^1$, which implies that $\text{ev}_p(\vec{B})$ is not stable.

This example also illustrates the fact that *there exist \mathbb{C} -semistable solutions of (6-8) which are not \mathbb{C} -stable*.

Fortunately, the existence of regular solutions for the real ADHM equations (1) and (2) for $r \geq 2$ can be used to guarantee the existence of \mathbb{C} -stable solutions of the complex equations.

Indeed, note that if V and W are provided with a Hermitian inner product, then the space of complex ADHM data \mathbb{B} acquires a natural involution $\dagger : \mathbb{B} \rightarrow \mathbb{B}$ given by:

$$\dagger(B_{11}, B_{12}, B_{21}, B_{22}, i_1, i_2, j_1, j_2) = (B_{22}^\dagger, -B_{21}^\dagger, -B_{12}^\dagger, B_{11}^\dagger, j_2^\dagger, -j_1^\dagger, -i_2^\dagger, i_1^\dagger)$$

The point $\vec{B} \in \mathbb{B}$ is said to be *real* if it is fixed by \dagger .

Note that if \vec{B} is real, then complex ADHM equations (6-8) above reduce to the usual ADHM equations with $\xi = 0$, by setting $B_1 = B_{11} = B_{22}^\dagger$, $B_2 = B_{12} = -B_{21}^\dagger$, $i = i_1 = j_2^\dagger$ and $j = j_1 = -i_2^\dagger$.

Proposition 6. *If (B_1, B_2, i, j) is a stable (hence regular) solution of (1) and (2) with $\xi = 0$, then $\vec{B} = (B_1, B_2, -B_2^\dagger, B_1^\dagger, i, -j^\dagger, j, i^\dagger)$ is a \mathbb{C} -regular solution of (6-8).*

In particular, it follows that $\mathcal{M}_{\mathbb{C}}(r, c)$ is non-empty for all $r \geq 2$ and for all $c \geq 1$. Moreover, there is a holomorphic surjective map:

$$\varepsilon : \mathcal{M}_{\mathbb{C}}^{\text{reg}}(r, c) \rightarrow \mathcal{M}^{\text{reg}}(r, c) \quad , \quad \forall r \geq 2, k \geq 1$$

$$\varepsilon(B_{kl}, i_k, j_k) := (B_{11}, B_{12}, i_1, j_1)$$

Clearly, ε is surjective, and the fibers $\varepsilon^{-1}(B_1, B_2, i, j)$ are closed subsets of $\mathcal{M}^{\text{reg}}(r, c)$ of dimension $2rc$.

Proof. It is easy to see that (B_1, B_2, i, j) satisfies (1) and (2) with $\xi = 0$, if and only if \vec{B} as above satisfies (6-8). Now note that in this case:

$$\tilde{B}_1 = zB_1 - wB_2^\dagger \quad , \quad \tilde{B}_2 = zB_2 + wB_1^\dagger \quad , \quad \tilde{i} = zi - wj^\dagger \quad .$$

If \vec{B} is not \mathbb{C} -stable, there is $[z : w] \in \mathbb{P}^1$ and a proper subspace $S \subset V$ such that $[\tilde{B}_1^\dagger, \tilde{B}_2^\dagger]|_S = 0$ and $S \subset \ker \tilde{i}^\dagger$. Thus $i^\dagger|_S = k \cdot j|_S$ for some $k \in \mathbb{C}$, hence $ii^\dagger|_S = k \cdot ij|_S = [B_1, B_2]|_S$. Hence $\text{Tr}(ii^\dagger|_S) = 0$, so that $[B_1^\dagger, B_2^\dagger]|_S = 0$ and $S \subset \ker i^\dagger$ and (B_1, B_2, i, j) is not stable.

Thus we conclude that \vec{B} as above is \mathbb{C} -stable. However, it is not difficult to see that every real, \mathbb{C} -stable complex ADHM datum is \mathbb{C} -regular. Indeed, if \vec{B} is real, then:

$$\begin{aligned} \tilde{B}_1 &= zB_{11} - wB_{12}^\dagger & \tilde{B}_2 &= zB_{12} + wB_{11}^\dagger \\ \tilde{i} &= zi_1 - wj_1^\dagger & \tilde{j} &= zj_1 + wi_1^\dagger \end{aligned}$$

Thus if $(\tilde{B}_1, \tilde{B}_2, \tilde{j})$ is not costable at $[z : w]$, then $(\tilde{B}_1, \tilde{B}_2, \tilde{i})$ is not stable at $[-\bar{w} : \bar{z}]$. \square

Remark 7. It is interesting to note that, differently from the real ADHM equations, *there are \mathbb{C} -stable solutions of (6-8) which are not \mathbb{C} -semiregular* (compare with Proposition 37). Indeed, for $r = 2$ and $c = 1$, we can take:

$$B_{kl} = 0 \quad , \quad i_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad i_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad j_1 = j_2 = 0 \quad .$$

Furthermore, *there are \mathbb{C} -semiregular solutions of (6-8) which are not \mathbb{C} -regular*; for $r = 3$ and $c = 1$, we can take:

$$B_{kl} = 0 \quad , \quad i_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad , \quad i_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad j_1 = (0 \ 0 \ 1) \quad , \quad j_2 = 0 \quad .$$

However, as we shall see in Section 3, every \mathbb{C} -semiregular solution of (6-8) for $r = 2$ is in fact \mathbb{C} -regular.

Our first main result follows from Proposition 6 together with Propositions 1 and 2:

Theorem 8. *$\mathcal{M}_{\mathbb{C}}(r, c)$ is a smooth complex manifold of dimension $4rc$, non-empty for $r \geq 2$, $c \geq 1$.*

Given the close analogy between $\mathcal{M}_{\mathbb{C}}(r, c)$ and the varieties $\mathcal{M}^{\text{reg}}(r, c)$ and $\mathcal{M}(r, c)$, we expect $\mathcal{M}_{\mathbb{C}}(r, c)$ to be a connected, hyperkähler quasi-projective algebraic variety for all $r \geq 2$ and $c \geq 1$. Indeed, one has for each $p \in \mathbb{P}^1$ a holomorphic map $\varepsilon_p : \mathcal{M}_{\mathbb{C}}(r, c) \rightarrow \mathcal{M}(r, c)$ induced by the natural evaluation map $\text{ev}_p : \mathbb{B} \rightarrow \mathbf{B}$ introduced above. One can then try to establish various properties of $\mathcal{M}_{\mathbb{C}}(r, c)$ by studying various properties (e.g. surjectivity) of the evaluation map.

1.4 Solutions for $c = 1$

For $c = \dim V = 1$, the varieties $\mathcal{M}_{\mathbb{C}}(r, c)$ can be described quite concretely. In this case, B_{kl} are just complex numbers, while i_k and j_k can be regarded as vectors in W ; the complex ADHM equations reduce to:

$$i_1 j_1 = i_2 j_2 = i_1 j_2 + i_2 j_1 = 0 \quad . \tag{11}$$

It is then easy to see that $\mathcal{M}_{\mathbb{C}}(r, 1) = \mathbb{C}^4 \times \mathcal{B}(r)$, where $\mathcal{B}(r)$ is a smooth quasi-projective variety of dimension $4(r - 1)$, described as follows.

Setting:

$$i_1 = (x_1 \ \cdots \ x_r) \quad i_2 = (y_1 \ \cdots \ y_r)$$

$$j_1 = \begin{pmatrix} z_1 \\ \vdots \\ z_r \end{pmatrix} \quad j_2 = \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix}$$

the equations (11) become:

$$\sum_{k=1}^r x_k z_k = \sum_{k=1}^r y_k w_k = \sum_{k=1}^r x_k w_k + y_k z_k = 0 . \quad (12)$$

while \mathbb{C} -stability is equivalent to the vectors (x_1, \dots, x_r) and (y_1, \dots, y_r) being linearly independent.

Then $\mathcal{B}(r)$ is the complete intersection of the three quadrics (12) within the open subset of $\mathbb{P}^{4r-1} = \mathbb{P}(W^{\oplus 4})$ consisting of \mathbb{C} -stable points. In particular, we conclude that $\mathcal{M}_{\mathbb{C}}(r, 1)$ is quasi-projective for all $r \geq 2$, in partial support of our general conjecture.

2 Sheaves on \mathbb{P}^3

In this section we will characterize $\mathcal{M}_{\mathbb{C}}(r, c)$ as a moduli space of certain sheaves on \mathbb{P}^3 . First, we recall the following definition, due to Manin [8].

Definition. *A coherent sheaf \mathcal{E} on \mathbb{P}^3 is said to be admissible if $H^p(\mathbb{P}^3, \mathcal{E}(k)) = 0$ for $p \leq 1$, $p + k \leq -1$ and for $p \geq 2$, $p + k \geq 0$.*

This somewhat mysterious cohomological condition is made natural once we recall that, under Penrose transform, locally-free admissible sheaves on \mathbb{P}^3 correspond to (framed) $GL(r, \mathbb{C})$ instantons (see Section 2.3 below). With a few extra assumptions on \mathcal{E} , the admissibility condition becomes a lot simpler; let $\ell_{\infty} = \{z = w = 0\}$.

Proposition 9. *Let \mathcal{E} be a torsion-free sheaf on \mathbb{P}^3 such that $\mathcal{E}|_{\ell_{\infty}} \simeq \mathcal{O}_{\ell_{\infty}}^{\oplus r}$. \mathcal{E} is admissible if and only if $H^1(\mathbb{P}^3, \mathcal{E}(-2)) = H^2(\mathbb{P}^3, \mathcal{E}(-2)) = 0$.*

Proof. Let \wp be a plane containing ℓ_{∞} , e.g. $\wp = \{z = 0\}$. Then $\mathcal{E}|_{\wp}$ is a torsion-free sheaf on \wp which is trivial at ℓ_{∞} . From [10] we know that:

$$H^0(\wp, \mathcal{E}|_{\wp}(k)) = 0 \quad \forall k \leq -1 \quad \text{and} \quad H^2(\wp, \mathcal{E}|_{\wp}(k)) = 0 \quad \forall k \geq -2 . \quad (13)$$

Now consider the sheaf sequence:

$$0 \rightarrow \mathcal{E}(k-1) \xrightarrow{\cdot z} \mathcal{E}(k) \rightarrow \mathcal{E}|_{\varphi}(k) \rightarrow 0 \quad (14)$$

Using (13), we conclude that:

$$H^3(\mathbb{P}^3, \mathcal{E}(k)) = H^3(\mathbb{P}^3, \mathcal{E}(k-1)) \quad \forall k \geq -2$$

But, by Serre's vanishing theorem, $H^3(\mathbb{P}^3, \mathcal{E}(N)) = 0$ for sufficiently large N , thus $H^3(\mathbb{P}^3, \mathcal{E}(k)) = 0$ for all $k \geq -3$.

Similarly, we have:

$$H^0(\mathbb{P}^3, \mathcal{E}(k-1)) = H^0(\mathbb{P}^3, \mathcal{E}(k)) \quad \forall k \leq -1$$

Since $\mathcal{E} \hookrightarrow \mathcal{E}^{**}$, we have via Serre duality:

$$H^0(\mathbb{P}^3, \mathcal{E}(k)) \hookrightarrow H^0(\mathbb{P}^3, \mathcal{E}^{**}(k)) = H^3(\mathbb{P}^3, \mathcal{E}^{***}(-k-4))^* .$$

Thus, again by Serre's vanishing theorem, $H^0(\mathbb{P}^3, \mathcal{E}(-N)) = 0$ for sufficiently large N , so that $H^0(\mathbb{P}^3, \mathcal{E}(k)) = 0$ for all $k \leq -1$.

We also have that:

$$0 \rightarrow H^1(\mathbb{P}^3, \mathcal{E}(k-1)) \rightarrow H^1(\mathbb{P}^3, \mathcal{E}(k)) \quad \forall k \leq -1$$

hence $H^1(\mathbb{P}^3, \mathcal{E}(-2)) = 0$ implies by induction that $H^1(\mathbb{P}^3, \mathcal{E}(k)) = 0$ for all $k \leq -2$. Furthermore,

$$H^2(\mathbb{P}^3, \mathcal{E}(k-1)) \rightarrow H^2(\mathbb{P}^3, \mathcal{E}(k)) \rightarrow 0 \quad \forall k \geq -2$$

forces $H^2(\mathbb{P}^3, \mathcal{E}(k)) = 0$ for all $k \geq -2$ once $H^2(\mathbb{P}^3, \mathcal{E}(-2)) = 0$. \square

In this section we will concentrate on *framed admissible torsion-free sheaves*, that is a pair (\mathcal{E}, ϕ) consisting of an admissible torsion-free sheaf such that the restriction $\mathcal{E}|_{\ell_\infty}$ is trivial plus a framing $\phi : \mathcal{E}|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus \text{rk} \mathcal{E}}$. We will show that the moduli space of framed admissible torsion-free sheaves can be parameterized by \mathbb{C} -stable complex ADHM data. More about admissible sheaves in general can be found at [7].

2.1 From ADHM data to sheaves

Let (B_{kl}, i_k, j_k) be a complex ADHM datum; combining constructions of Donaldson [3] and Nakajima [10], we define the monad:

$$V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^3}(1) \quad (15)$$

where the maps α and β are given by:

$$\alpha = \begin{pmatrix} zB_{11} + wB_{21} + x \\ zB_{12} + wB_{22} + y \\ zj_1 + wj_2 \end{pmatrix} \quad (16)$$

$$\beta = \begin{pmatrix} -zB_{12} - wB_{22} - y & zB_{11} + wB_{21} + x & zi_1 + wi_2 \end{pmatrix} \quad (17)$$

Proposition 10. $\beta\alpha = 0$ if and only if (B_{kl}, i_k, j_k) satisfies the complex ADHM equations (6-8).

The proof is a straightforward calculation left to the reader. It follows from Proposition 10 that $\mathcal{E} = \ker \beta / \text{Im} \alpha$, the first cohomology of the monad (15), is a well-defined coherent sheaf on \mathbb{P}^3 . We will now check that \mathcal{E} is the only nontrivial cohomology of (15). It is easy to see that $GL(V)$ -equivalent complex ADHM data will produce isomorphic cohomology sheaves.

Proposition 11. α_X is injective away from a subvariety of codimension 2.

In particular, α is injective as a sheaf map.

Proof. It is easy to see that α is injective on the line $\ell_\infty = \{z = w = 0\}$. So consider a point $X = [x : y : z : w] \in \mathbb{P}^3 \setminus \ell_\infty$, and take $v \in V$ such that $\alpha_X(v) = 0$, that is:

$$\begin{cases} (zB_{11} + wB_{21})v = -xv \\ (zB_{12} + wB_{22})v = -yv \\ (zj_1 + wj_2)v = 0 \end{cases} \quad (18)$$

Thus v is a common eigenvector of $zB_{11} + wB_{21}$ and $zB_{12} + wB_{22}$, with eigenvalues $-x$ and $-y$, respectively. Hence, for fixed $z, w \neq 0$, we conclude

that x and y may assume only finitely many values. In other words, α_X may fail to be injective only at points of the form $[x = x(z, w) : y = y(z, w) : z : w]$, so α_X is injective away from a codimension 2 subvariety (which does not intersect ℓ_∞). \square

The following is the key result in the monad construction, and further justifies our concept of \mathbb{C} -stability:

Proposition 12. *β is surjective if and only if (B_{kl}, i_k, j_k) is \mathbb{C} -stable.*

Proof. Again, it is easy to see that β is surjective on the line $\ell_\infty = \{z = w = 0\}$. So it is enough to show that the localization of β to all points $X = [x : y : z : w] \in \mathbb{P}^3 \setminus \ell_\infty$ is surjective.

Equivalently, we argue that if (B_{kl}, i_k, j_k) is \mathbb{C} -stable, then the dual map β_X^* is injective for all $X \in \mathbb{P}^3 \setminus \ell_\infty$. Indeed, β_X^* is not injective for some $[x : y : p_1 : p_2]$, then there is $v \in V$ such that:

$$\begin{cases} \tilde{B}_1(p_1, p_2)^*v = \bar{x}v \\ \tilde{B}_2(p_1, p_2)^*v = -\bar{y}v \\ \tilde{i}(p_1, p_2)^*v = 0 \end{cases} \quad (19)$$

which, by duality, implies that $(\tilde{B}_1(p_1, p_2), \tilde{B}_2(p_1, p_2), \tilde{i}(p_1, p_2))$ is not stable. Thus (B_{kl}, i_k, j_k) is not \mathbb{C} -stable.

The converse statement is now clear: if (B_{kl}, i_k, j_k) is not \mathbb{C} -stable, then by duality β_X^* is not injective for some $[x : y : z : w]$, hence β cannot be surjective. \square

In order to further characterize the cohomology sheaf $\mathcal{E} = \ker \beta / \text{Im} \alpha$, let $[H]$ denote the generator of $H^\bullet(\mathbb{P}^3, \mathbb{C})$, i.e. $[H] = c_1(\mathcal{O}_{\mathbb{P}^3}(1))$.

Proposition 13. *The cohomology sheaf $\mathcal{E} = \ker \beta / \text{Im} \alpha$ is an admissible torsion-free sheaf on \mathbb{P}^3 , with $\text{ch}(\mathcal{E}) = r - c[H]^2$. Moreover, $\mathcal{E}|_{\ell_\infty}$ is trivial.*

Proof. For the admissibility of \mathcal{E} , set $\mathcal{K} = \ker \beta$. From the sequence

$$0 \rightarrow \mathcal{K} \rightarrow \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0 \quad (20)$$

we obtain, after tensoring (20) by $\mathcal{O}_{\mathbb{P}^3}(k)$, the exact sequence of cohomology:

$$V \otimes H^{p-1}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k+1)) \rightarrow H^p(\mathbb{P}^3, \mathcal{K}(k)) \rightarrow \tilde{W} \otimes H^p(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k))$$

which implies that $H^p(\mathbb{P}^3, \mathcal{K}(k)) = 0$ for $p \leq 1$, $p+k \leq -1$ and for $p \geq 2$, $p+k \geq 0$, since the two groups at the ends are zero in this range. Next, from the sequence:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{K} \rightarrow \mathcal{E} \rightarrow 0 \quad (21)$$

we obtain, after tensoring (20) by $\mathcal{O}_{\mathbb{P}^3}(k)$, the exact sequence of cohomology:

$$H^p(\mathbb{P}^3, \mathcal{K}(k)) \rightarrow H^p(\mathbb{P}^3, \mathcal{E}(k)) \rightarrow V \otimes H^{p+1}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-1))$$

and hence $H^p(\mathbb{P}^3, \mathcal{E}(k)) = 0$ for $p \leq 1$, $p+k \leq -1$ and for $p \geq 2$, $p+k \geq 0$, since the third group also vanishes in that range.

To compute $\text{ch}(\mathcal{E})$, just notice that:

$$\text{ch}(\mathcal{E}) = \text{ch}(\tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}) - \text{ch}(V \otimes \mathcal{O}_{\mathbb{P}^3}(-1)) - \text{ch}(V \otimes \mathcal{O}_{\mathbb{P}^3}(1))$$

since \mathcal{E} is the only non vanishing cohomology of the monad (15). The triviality of $\mathcal{E}|_{\ell_\infty}$ also follows easily from the construction, see [3].

It remains for us to show that \mathcal{E} is torsion-free. First, notice that \mathcal{K} is a locally-free sheaf, since β_X is surjective for all $X \in \mathbb{P}^3$ (see the proof of Proposition 12). Moreover, as it was pointed out in the proof of Proposition 11, the α_X is injective away from a subset of codimension 2 in \mathbb{P}^3 . Applying Proposition 40 in Appendix C to sequence (21), we conclude that \mathcal{E} must be torsion-free. \square

We are finally in position to prove Proposition 5 by looking at the corresponding sheaves on \mathbb{P}^3 .

Proposition 14. *There are no admissible torsion-free sheaves \mathcal{E} on \mathbb{P}^3 with $\text{ch}(\mathcal{E}) = 1 - c[H]^2$.*

Indeed, if there were \mathbb{C} -stable solutions of (6-8) for $r = 1$, the monad construction would produce admissible torsion-free sheaves \mathcal{E} such that $\text{ch}(\mathcal{E}) = 1 - c[H]^2$. So the above result implies Proposition 5.

Proof. First note that $\mathcal{E}^{**} \simeq \mathcal{O}_{\mathbb{P}^3}$, since \mathcal{E}^{**} is reflexive of rank 1 (hence locally-free) and $c_1(\mathcal{E}^{**}) = c_1(\mathcal{E}) = 0$. So \mathcal{E} is an ideal sheaf fitting in the sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{Q} \rightarrow 0 \quad , \quad \mathcal{Q} = \mathcal{O}_{\mathbb{P}^3}/\mathcal{E} . \quad (22)$$

Note that $\text{ch}(\mathcal{Q}) = c[H]^2$, and that \mathcal{Q} is the structure sheaf of a 1-dimensional subscheme $s : \Sigma \hookrightarrow \mathbb{P}^3$ (i.e. $\mathcal{Q} = s_*\mathcal{O}_\Sigma$).

After twisting the sequence (22) by $\mathcal{O}_{\mathbb{P}^3}(k)$ and using admissibility, we get that $h^0(\mathbb{P}^3, \mathcal{Q}(k)) = 0$ for all $k \leq -2$, hence the $\Sigma = \text{supp } \mathcal{Q}$ contains no 0-dimensional components. Moreover, $h^1(\mathbb{P}^3, \mathcal{Q}(k)) = 0$ for all $k \geq -2$; in particular

$$h^0(\Sigma, \mathcal{O}_\Sigma) = h^0(\mathbb{P}^3, \mathcal{Q}) = \chi(\mathbb{P}^3, \mathcal{Q}) = 2c ,$$

so that Σ consists of $2c$ connected components $\Sigma_1, \dots, \Sigma_{2c}$.

For each connected component Σ_a , we have that $\chi(\mathbb{P}^3, \mathcal{O}_{\Sigma_a}) = \chi(\Sigma_a, \mathcal{O}_{\Sigma_a}) = 1$. It follows that Σ_a is a smooth \mathbb{P}^1 , so \mathcal{E} must be the ideal sheaf of $2c$ lines in \mathbb{P}^3 . But it is easy to check that the Chern character of the ideal sheaf of $2c$ lines in \mathbb{P}^3 is given by $1 - 2c[H]^2 + 2c[H]^3$, leading to a contradiction. \square

2.2 From sheaves to complex ADHM data

The first step of the reverse construction is essentially provided by Manin [8]:

Proposition 15. *Every admissible torsion-free sheaf \mathcal{E} on \mathbb{P}^3 can be obtained as the cohomology of the monad*

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3} \xrightarrow{\beta} V' \otimes \mathcal{O}_{\mathbb{P}^3}(1) \rightarrow 0 , \quad (23)$$

where $V = H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1))$, $\tilde{W} = H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1)$ and $V' = H^1(\mathcal{E}(-1))$.

Proof. Manin proves the case \mathcal{E} being locally-free in [8, p. 91], using the Beilinson spectral sequence. However, the argument generalizes word by word for \mathcal{E} being torsion-free; just note that the projection formula

$$R^i p_{1*}(p_1^* \mathcal{O}_{\mathbb{P}^3}(k) \otimes p_2^* \mathcal{F}) = \mathcal{O}_{\mathbb{P}^3}(k) \otimes H^i(\mathcal{F})$$

holds for every torsion-free sheaf \mathcal{F} , where p_1 and p_2 are the natural projections of $\mathbb{P}^3 \times \mathbb{P}^3$ onto the first and second factors. \square

So let \mathcal{E} be an admissible torsion-free sheaf on \mathbb{P}^3 with $\text{ch}(\mathcal{E}) = r - c[H]^2$ and such that $\mathcal{E}|_{\ell_\infty}$ is trivial. It remains for us to show that the monad in Proposition 15 can be reduced to a \mathbb{C} -stable solution of the complex ADHM equations (6-8). A lengthy but straightforward cohomological calculation (see Appendix B) shows that:

$$h^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) = h^1(\mathbb{P}^3, \mathcal{E}(-1)) = c, \quad h^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = c + 2r$$

and that there is a natural identification $V \simeq V'$.

Now, $\alpha \in \text{Hom}(V, \tilde{W}) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ and $\beta \in \text{Hom}(\tilde{W}, V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$, and we can express these maps in the following manner:

$$\alpha = \alpha_1 x + \alpha_2 y + \alpha_3 z + \alpha_4 w \quad \text{and} \quad \beta = \beta_1 x + \beta_2 y + \beta_3 z + \beta_4 w$$

where, clearly, $\alpha_k \in \text{Hom}(V, \tilde{W})$ and $\beta_k \in \text{Hom}(\tilde{W}, V)$ for each $k = 1, \dots, 4$. The condition $\beta\alpha = 0$ then implies that:

$$\beta_k \alpha_k = 0, \quad k = 1, \dots, 4$$

$$\beta_k \alpha_l + \beta_l \alpha_k = 0, \quad k, l = 1, \dots, 4 \text{ and } k \neq l$$

Restricting (23) to the line at infinity $\ell_\infty = \{z = w = 0\}$ we get:

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(-1) \xrightarrow{\alpha_\infty} \tilde{W} \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty} \xrightarrow{\beta_\infty} V \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(1) \rightarrow 0$$

where $\alpha_\infty = \alpha_1 x + \alpha_2 y$ and $\beta_\infty = \beta_1 x + \beta_2 y$. Setting $\mathcal{K} = \ker \beta$ we have:

$$0 \rightarrow V \otimes \mathcal{O}_{\ell_\infty}(-1) \xrightarrow{\alpha_\infty} \mathcal{K}|_{\ell_\infty} \longrightarrow \mathcal{E}|_{\ell_\infty} \rightarrow 0$$

from the associated long exact sequence of cohomology we conclude that $H^1(\ell_\infty, \mathcal{K}|_{\ell_\infty}) = 0$ and $H^0(\ell_\infty, \mathcal{K}|_{\ell_\infty}) \simeq H^0(\ell_\infty, \mathcal{E}|_{\ell_\infty}) \simeq \mathcal{E}_P$, for some $P \in \ell_\infty$, since $H^p(\ell_\infty, \mathcal{O}_{\ell_\infty}(-1)) = 0$, for $p = 1, 2$, and since $\mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r}$. We set $W = H^0(\ell_\infty, \mathcal{K}|_{\ell_\infty})$; the choice of a basis for W corresponds to the choice of a trivialization for $\mathcal{E}|_{\ell_\infty}$.

Similarly, from the sequence

$$0 \rightarrow \mathcal{K}|_{\ell_\infty} \longrightarrow \tilde{W} \otimes \mathcal{O}_{\ell_\infty} \xrightarrow{\beta_\infty} V \otimes \mathcal{O}_{\ell_\infty}(1) \rightarrow 0$$

we obtain:

$$0 \rightarrow W \longrightarrow \tilde{W} \xrightarrow{\beta_\infty} V \otimes H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \rightarrow 0 \quad (24)$$

since $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}) \simeq \mathbb{C}$ and $H^1(\ell_\infty, \mathcal{K}|_{\ell_\infty}) = 0$. Then using the identification $H^0(\ell_\infty, \mathcal{O}_{\ell_\infty}(1)) \simeq \mathbb{C}x \oplus \mathbb{C}y$ we can rewrite (24) in the following way:

$$0 \rightarrow W \longrightarrow \tilde{W} \xrightarrow{\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}} V \oplus V \rightarrow 0 \quad (25)$$

so that $W = \ker \beta_1 \cap \ker \beta_2$.

Applying the same argument to the dual monad:

$$0 \rightarrow V^* \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(-1) \xrightarrow{\beta_\infty^t} \tilde{W}^* \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty} \xrightarrow{\alpha_\infty^t} V^* \otimes \mathcal{O}_{\mathbb{P}^3}|_{\ell_\infty}(1) \rightarrow 0$$

we have the exact sequence:

$$0 \rightarrow H^0(\ell_\infty, \ker\{\alpha_\infty^t\}) \longrightarrow \tilde{W}^* \xrightarrow{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}} V^* \oplus V^*$$

which implies that $(\alpha_1 \ \alpha_2) : V \oplus V \rightarrow \tilde{W}$ is injective. Moreover, the sequence (25) splits, and we can identify $\tilde{W} \simeq V \oplus V \oplus W$.

Furthermore, notice that

$$\ker \beta_1 / \text{Im} \alpha_1 \simeq \mathcal{E}_{[1,0,0,0]} \simeq W \simeq \ker \beta_1 \cap \ker \beta_2 .$$

Thus $\text{Im} \alpha_1 \cap \ker \beta_2 = 0$, so that $\beta_1 \alpha_2 = -\beta_2 \alpha_1 : V \rightarrow V$ are isomorphisms.

Therefore we have:

$$\alpha_1 = \begin{pmatrix} \mathbf{1}_V \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ \mathbf{1}_V \\ 0 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 0 & \mathbf{1}_V & 0 \end{pmatrix} \\ \beta_2 = \begin{pmatrix} -\mathbf{1}_V & 0 & 0 \end{pmatrix}$$

and the condition $\beta\alpha = 0$ implies that:

$$\alpha_3 = \begin{pmatrix} B_{11} \\ B_{12} \\ j_1 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} B_{21} \\ B_{22} \\ j_2 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} -B_{12} & B_{11} & i_1 \end{pmatrix} \\ \beta_4 = \begin{pmatrix} -B_{22} & B_{21} & i_2 \end{pmatrix}$$

with (B_{kl}, i_k, j_k) being a complex ADHM datum satisfying the complex ADHM equations (6-8). The surjectivity of β implies the \mathbb{C} -stability of (B_{kl}, i_k, j_k) , by Proposition 12. Summing up, we have proved:

Theorem 16. *There is a 1-1 correspondence between the following objects:*

- *framed admissible torsion-free sheaves on \mathbb{P}^3 , and*
- *\mathbb{C} -stable solutions of the complex ADHM equations.*

In particular, the moduli space of framed admissible torsion-free sheaves \mathcal{E} on \mathbb{P}^3 with $\text{ch}(\mathcal{E}) = r - c[H]^2$ is a smooth complex manifold of dimension $4rc$, non-empty for $r \geq 2$.

2.3 Locally-free admissible sheaves and complex instantons

We will now describe necessary and sufficient conditions that guarantee that the cohomology sheaf of the monad (15) is locally-free. Recall that α_X denotes the localization of the map α to a point $X \in \mathbb{P}^3$.

Proposition 17. *α_X is injective for all $X \in \mathbb{P}^3$ if and only if (B_{kl}, i_k, j_k) is \mathbb{C} -costable.*

Proof. Since α is injective on the line $\ell_\infty = \{z = w = 0\}$, it is enough to show that α_X is injective for all $X = [x : y : z : w] \in \mathbb{P}^3 \setminus \ell_\infty$.

Indeed, take $v \in V$ such that $\alpha_X(v) = 0$, hence:

$$\begin{cases} \tilde{B}_1 v = -xv \\ \tilde{B}_2 v = -yv \\ \tilde{j}v = 0 \end{cases}$$

But $(\tilde{B}_k, \tilde{j}, \tilde{j})$ is costable for all $(z, w) \in \mathbb{C}^2 \setminus \{0\}$, therefore $v = 0$.

Conversely, if (B_{kl}, i_k, j_k) is not \mathbb{C} -costable, there are $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$ and a proper subspace $S \subset V$ such that $[\tilde{B}_1, \tilde{B}_2]|_S = 0$ and $S \subset \ker \tilde{j}$. Therefore $\alpha_{[x:y:\lambda:\mu]}(v) = 0$ for all $v \in S$. \square

Thus if $\vec{B} = (B_{kl}, i_k, j_k)$ is \mathbb{C} -regular, then α_X is injective and β_X is surjective for all $X \in \mathbb{P}^3$, so that the quotient $\ker \beta_X / \text{Im} \alpha_X$ is a vector space of dimension r for all $X \in \mathbb{P}^3$. We conclude that:

Corollary 18. *The cohomology sheaf \mathcal{E} is locally-free if and only if (B_{kl}, i_k, j_k) is \mathbb{C} -regular.*

Remark 19. As it was pointed out in Remark 7, there are solutions of the complex ADHM equations which are \mathbb{C} -regular but not \mathbb{C} -stable. Therefore, *there exist admissible torsion-free sheaves which are not locally-free.* The basic example is the cohomology \mathcal{E} of the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)$$

$$\alpha = \begin{pmatrix} x \\ y \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} -y & x & z & w \end{pmatrix}$$

It is easy to see that β is surjective for all $(x, y, z, w) \in \mathbb{P}^3$, while α is injective provided $x, y \neq 0$. It then follows from applying Proposition 40 to sequence (21) that \mathcal{E} is torsion-free, but not locally-free. In particular, the singularity set of \mathcal{E} (i.e. the support of $\mathcal{E}^{**}/\mathcal{E}$) consists of the line $\{x = y = 0\} \subset \mathbb{P}^3$.

Rank 2 locally-free sheaves \mathcal{E} with $c_1(\mathcal{E}) = 0$ and $H^0(\mathbb{P}^3, E(-1)) = H^1(\mathbb{P}^3, \mathcal{E}(-2)) = 0$ are also known in the literature as *mathematical* (or *complex*) *instanton bundles* (see [12, 16] and also [2] for more recent references and a brief survey of the subject); it is easy to see, via Serre duality, that these are admissible. They correspond, via the Penrose transform, to holomorphic vector bundles with $SL(2, \mathbb{C})$ anti-self-dual connections on \mathbb{M} , the complexified compactified Minkowski space-time (see [16]; recall also that \mathbb{M} is just the Grassmannian of lines in \mathbb{P}^3). The integer $c = c_2(\mathcal{E})$ is also called the *charge* of the complex instanton bundle \mathcal{E} .

For rank $r > 2$, a complex instanton bundle is an admissible locally-free sheaf and a framed complex instanton bundle is a framed admissible locally-free sheaf. Clearly, the moduli space of equivalence classes of framed complex instanton bundles fibers over the moduli space of equivalence classes of complex instanton bundles, with fibers given by $PGL(r, \mathbb{C})$, the set of all possible framings. It follows that the moduli space of framed complex instanton bundles of rank $r \geq 2$ and charge $c \geq 1$ is exactly $\mathcal{M}_{\mathbb{C}}^{\text{reg}}(r, c)$, the open subset of $\mathcal{M}_{\mathbb{C}}(r, c)$ consisting of the orbits of \mathbb{C} -regular solutions of the complex ADHM equations (6-8).

Determining the irreducibility and smoothness of the moduli space of rank 2 complex instanton bundles with charge c is a long standing question, see [2] for a recent short survey of this topic. As a special case of Theorem 16, we obtain a strong result along these lines:

Corollary 20. *The moduli space of framed complex instanton bundles of rank $r \geq 2$ and charge c is a nonempty, smooth complex manifold of dimension $4rc$.*

We remark that framed complex instanton bundles are always μ -semistable, see [12, p. 210]. It would be interesting to compare the admissibility and semistability condition, and determine under what necessary and sufficient conditions admissible torsion-free sheaves are μ -semistable, and vice-versa; a few results along these lines have been obtained by the first author in [7].

2.4 Reflexive admissible sheaves

Reflexive sheaves on \mathbb{P}^3 have been extensively studied in a series of papers by Hartshorne [5], among other authors. In particular, it was shown that a rank 2 reflexive sheaf on \mathbb{P}^3 is locally-free if and only if $c_3(\mathcal{F}) = 0$. Therefore, we conclude:

Proposition 21. (Hartshorne [5]) *There are no rank 2 admissible sheaves on \mathbb{P}^3 which are reflexive but not locally-free.*

The situation for higher rank is quite different, though, and it is easy to construct a rank 3 admissible sheaf which is reflexive but not locally-free. Setting $r = 3$ and $c = 1$, consider the monad:

$$\mathcal{O}_{\mathbb{P}^3}(-1) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}^{\oplus 5} \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(1)$$

$$\alpha = \begin{pmatrix} x \\ y \\ 0 \\ 0 \\ z \end{pmatrix} \quad \text{and} \quad \beta = (-y \ x \ z \ w \ 0)$$

Again, it is easy to see that β is surjective for all $(x, y, z, w) \in \mathbb{P}^3$, while α is injective provided $x, y, z \neq 0$. It then follows from applying Proposition 40 to sequence (21) that \mathcal{E} is reflexive, but not locally-free; its singularity set is just the point $[0 : 0 : 0 : 1] \in \mathbb{P}^3$.

Proposition 22. *The cohomology sheaf \mathcal{E} is reflexive if and only if \vec{B} is \mathbb{C} -semiregular.*

It then follows from Hartshorne's result that every \mathbb{C} -semiregular solution of the complex ADHM equations for $r = 2$ is \mathbb{C} -regular, as we claimed in Remark 7; the example of a properly reflexive admissible sheaf corresponds to the properly \mathbb{C} -semiregular solution of the complex ADHM equations for $r = 3$ given in Remark 7.

Proof. If \vec{B} is \mathbb{C} -semiregular, then \vec{B}_p is costable for all but finitely many $p \in \mathbb{P}^1$ (see the observation following Proposition 3). This means that α_X is injective for all but finitely many $X \in \mathbb{P}^3$. Thus, by Proposition 40, the cohomology sheaf \mathcal{E} is reflexive.

Conversely, if \mathcal{E} is reflexive then α_X is injective for all but finitely many $X \in \mathbb{P}^3$. It follows that \vec{B}_p must be costable for all but finitely many $p \in \mathbb{P}^1$, and \vec{B} is \mathbb{C} -semiregular. \square

3 Quantum instantons

In this section we will adapt the ADHM construction of instantons to obtain *complex quantum instantons* (cf. [4]) from \mathbb{C} -regular solutions of the complex ADHM equations. For the convenience of the reader, let us first recall the essential definitions from our previous paper.

3.1 Quantum Minkowski space-time

In [4], we defined the *quantum compactified, complexified Minkowski space* $\mathfrak{M}_{p,q}$ as the associative graded \mathbb{C} -algebra generated by $z_{11'}, z_{12'}, z_{21'}, z_{22'}, D, D'$ satisfying the relations (26) to (30) below ($p = q^{\pm 1}$ are formal parameters):

$$\begin{aligned} z_{11'} z_{12'} &= z_{12'} z_{11'} & z_{11'} z_{21'} &= z_{21'} z_{11'} \\ z_{12'} z_{22'} &= z_{22'} z_{12'} & z_{21'} z_{22'} &= z_{22'} z_{21'} \\ z_{12'} z_{21'} &= z_{21'} z_{12'} \end{aligned} \quad (26)$$

$$q^{-1}(z_{11'} z_{22'} - z_{12'} z_{21'}) = q(z_{22'} z_{11'} - z_{12'} z_{21'}) \quad (27)$$

$$\begin{aligned} Dz_{11'} &= pq^{-1} z_{11'} D & D' z_{11'} &= p^{-1} q^{-1} z_{11'} D' \\ Dz_{12'} &= pq^{-1} z_{12'} D & D' z_{12'} &= p^{-1} q z_{12'} D' \\ Dz_{21'} &= pq z_{21'} D & D' z_{21'} &= p^{-1} q^{-1} z_{21'} D' \\ Dz_{22'} &= pq z_{22'} D & D' z_{22'} &= p^{-1} q z_{22'} D' \end{aligned} \quad (28)$$

$$p^{-1} DD' = p D' D \quad (29)$$

$$q^{-1}(z_{11'} z_{22'} - z_{12'} z_{21'}) = p^{-1} DD' \quad (30)$$

Localizations of $\mathfrak{M}_{p,q}$ with respect to D and D' lead to two “affine patches” \mathfrak{M}_q^I and \mathfrak{M}_q^J , respectively. More precisely, the new generators:

$$x_{rs'} = \frac{z_{rs'}}{D} \quad \text{and} \quad y_{rs'} = \frac{z_{rs'}}{D'}$$

satisfy the relations:

$$x_{11'}x_{12'} = x_{12'}x_{11'}, \quad x_{21'}x_{22'} = x_{22'}x_{21'}, \quad (31)$$

$$[x_{11'}, x_{22'}] + [x_{21'}, x_{12'}] = 0 \quad (32)$$

$$\begin{aligned} x_{11'}x_{21'} &= q^{-2}x_{21'}x_{11'}, & x_{12'}x_{22'} &= q^{-2}x_{22'}x_{12'}, \\ x_{21'}x_{12'} &= q^2x_{12'}x_{21'} \end{aligned} \quad (33)$$

and

$$y_{11'}y_{21'} = y_{21'}y_{11'}, \quad y_{12'}y_{22'} = y_{22'}y_{12'}, \quad (34)$$

$$[y_{11'}, y_{22'}] + [y_{12'}, y_{21'}] = 0 \quad (35)$$

$$\begin{aligned} y_{11'}y_{12'} &= q^{-2}y_{12'}y_{11'}, & y_{21'}y_{22'} &= q^{-2}y_{22'}y_{21'}, \\ y_{12'}y_{21'} &= q^2y_{21'}y_{12'} \end{aligned} \quad (36)$$

$$\text{i.e. } \mathfrak{M}_q^I = \mathfrak{M}_{p,q}[D^{-1}] = \mathbb{C}[x_{11'}, x_{12'}, x_{21'}, x_{22'}]/(31-33) .$$

$$\text{and } \mathfrak{M}_q^J = \mathfrak{M}_{p,q}[D'^{-1}] = \mathbb{C}[y_{11'}, y_{12'}, y_{21'}, y_{22'}]/(34-36) .$$

These two algebras can be made isomorphic after inverting the determinants $\det(x) = x_{11'}x_{22'} - x_{12'}x_{21'}$ and $\det(y) = y_{11'}y_{22'} - y_{21'}y_{12'}$. Note that $\det(x) = \det(y)^{-1} = \frac{D'}{D}$. One has:

$$\mathfrak{M}_q^I[\det(x)^{-1}] \xrightarrow{\sim} \mathfrak{M}_q^J[\det(y)^{-1}] \quad (37)$$

$$\text{where } y_{kl'} = \frac{x_{kl'}}{\det(x)} .$$

We denote \mathfrak{M}_q^{IJ} the isomorphic algebras in (37).

Let us now focus on \mathfrak{M}_q^I ; all observations below will also apply to \mathfrak{M}_q^J .

It follows immediately from the commutation relations (31-33) that any element of \mathfrak{M}_q^I can be presented as a sum of monomials of the form:

$$x_{11'}^{n_{11'}} x_{12'}^{n_{12'}} x_{21'}^{n_{21'}} x_{22'}^{n_{22'}} \quad , \quad n_{ij'} \geq 0 \quad (38)$$

Moreover, it is easy to see directly from (31-33), and it is also proven in [4, Theorem 21], that these monomials are linearly independent and therefore form a basis of \mathfrak{M}_q^I . An element $f \in \mathfrak{M}_q^I$ has degree d if it is a sum of monomials (38) with $n_{11'} + n_{12'} + n_{21'} + n_{22'} = d$.

A concise form of the commutation relations (31-33) can also be expressed in terms of an R -matrix:

$$R_{12}^I = \begin{pmatrix} p^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} & p^{-1} - q & 0 \\ 0 & p^{-1} - q^{-1} & q & 0 \\ 0 & 0 & 0 & p^{-1} \end{pmatrix}, \quad p = q^{\pm 1} \quad (39)$$

Setting

$$X = \begin{pmatrix} x_{11'} & x_{12'} \\ x_{21'} & x_{22'} \end{pmatrix}$$

and defining $X_1 = X \otimes \mathbf{1}$ and $X_2 = \mathbf{1} \otimes X$, the relations (31-33) become equivalent to the identity:

$$R_{12}^I X_1 X_2 = X_2 X_1 R_{12}^I$$

We define the module of 1-forms over \mathfrak{M}_q^I , denoted by $\Omega_{\mathfrak{M}_q^I}^1$, as the \mathfrak{M}_q^I -bimodule generated by:

$$dX = \begin{pmatrix} dx_{11'} & dx_{12'} \\ dx_{21'} & dx_{22'} \end{pmatrix}$$

satisfying the following relations (written in matrix form):

$$R_{12}^I X_1 dX_2 = dX_2 X_1 (R_{21}^I)^{-1} \quad (40)$$

where $dX_2 = \mathbf{1} \otimes dX$ and $R_{21}^I = Q_1^{-1} Q_2 R_{21} Q_1^{-1} Q_2$

Similarly, the module of 2-forms $\Omega_{\mathfrak{M}_q^I}^2$ is the \mathfrak{M}_q^I -bimodule generated by $dx_{rs'} \wedge dx_{kl'}$ satisfying the relations (written in matrix form):

$$R_{12}^I dX_1 \wedge dX_2 = -dX_2 \wedge dX_1 (R_{21}^I)^{-1} \quad (41)$$

where $dX_1 = dX \otimes \mathbf{1}$. The module $\Omega_{\mathfrak{M}_q}^2$ splits as the sum of the submodules:

$$\Omega_{\mathfrak{M}_q}^{2,+} = \mathfrak{M}_q^1 \langle dx_{11'} \wedge dx_{12'}, dx_{21'} \wedge dx_{22'}, dx_{11'} \wedge dx_{22'} - dx_{12'} \wedge dx_{21'} \rangle \quad (42)$$

$$\Omega_{\mathfrak{M}_q}^{2,-} = \mathfrak{M}_q^1 \langle dx_{11'} \wedge dx_{21'}, dx_{12'} \wedge dx_{22'}, dx_{11'} \wedge dx_{22'} + dx_{12'} \wedge dx_{21'} \rangle \quad (43)$$

which can be regarded as the modules of self-dual and anti-self-dual 2-forms, respectively.

Finally, the action of the de Rham operator $d : \mathfrak{M}_q^1 \rightarrow \Omega_{\mathfrak{M}_q}^1$ is given on the generators as $x_{rs'} \mapsto dx_{rs'}$, and it is then extended to the whole \mathfrak{M}_q^1 by \mathbb{C} -linearity and the Leibnitz rule:

$$d(fg) = gdf + fdg \quad (44)$$

where $f, g \in \mathfrak{M}_q^1$. One also defines the de Rham operator $d : \Omega_{\mathfrak{M}_q}^1 \rightarrow \Omega_{\mathfrak{M}_q}^2$ on the generators as $fdx_{rs'} \mapsto df \wedge dx_{rs'}$, also extending it by \mathbb{C} -linearity and the Leibnitz rule (44). Relations (40) and (41) imply that $d^2 = 0$.

Now let E be a right \mathfrak{M}_q^1 -module. A *connection* on E is a \mathbb{C} -linear map:

$$\nabla : E \rightarrow E \otimes_{\mathfrak{M}_q^1} \Omega_{\mathfrak{M}_q^1}^1$$

satisfying the Leibnitz rule:

$$\nabla(\sigma f) = \sigma \otimes df + \nabla(\sigma) f$$

where $f \in \mathfrak{M}_q^1$ and $\sigma \in E$. The connection ∇ also acts on 1-forms, being defined as the \mathbb{C} -linear map:

$$\nabla : E \otimes_{\mathfrak{M}_q^1} \Omega_{\mathfrak{M}_q^1}^1 \rightarrow E \otimes_{\mathfrak{M}_q^1} \Omega_{\mathfrak{M}_q^1}^2$$

satisfying:

$$\nabla(\sigma \otimes \omega) = \sigma \otimes d\omega + \nabla\sigma \wedge \omega$$

where $\omega \in \Omega_{\mathfrak{M}_q^1}^1$.

Moreover, two connections ∇ and ∇' are said to be gauge equivalent if there is $g \in \text{Aut}_{\mathfrak{M}_q^1}(E)$ such that $\nabla = g^{-1}\nabla'g$.

The *curvature* F_∇ is defined by the composition:

$$E \xrightarrow{\nabla} E \otimes_{\mathfrak{M}_q^I} \Omega_{\mathfrak{M}_q^I}^1 \xrightarrow{\nabla} E \otimes_{\mathfrak{M}_q^I} \Omega_{\mathfrak{M}_q^I}^2$$

and it is easy to check that F_∇ is actually right \mathfrak{M}_q^I -linear. Therefore, F_∇ can be regarded as an element of $\text{End}_{\mathfrak{M}_q^I}(E) \otimes_{\mathfrak{M}_q^I} \Omega_{\mathfrak{M}_q^I}^2$. Furthermore, if ∇ and ∇' are gauge equivalent, then there is $g \in \text{Aut}_{\mathfrak{M}_q^I}(E)$ such that $F_\nabla = g^{-1} F_{\nabla'} g$. A connection ∇ is said to be anti-self-dual if $F_\nabla \in \text{End}_{\mathfrak{M}_q^I}(E) \otimes_{\mathfrak{M}_q^I} \Omega_{\mathfrak{M}_q^I}^{2,-}$.

Definition. A complex quantum instanton over $\mathfrak{M}_{p,q}$ consists of the following data:

1. finitely generated free right \mathfrak{M}_q^I - and \mathfrak{M}_q^J -modules E_I and E_J ;
2. anti-self-dual connections ∇_I and ∇_J on E_I and E_J , respectively;
3. an isomorphism $\Gamma : E_I[\det(x)^{-1}] \rightarrow E_J[\det(y)^{-1}]$ satisfying $\nabla_J \Gamma = \Gamma \nabla_I$.

The attentive reader will notice that in our previous paper [4] we defined a quantum instanton as a pair consisting of a projective module and an anti-self-dual connection. As we will see below, the modules produced via ADHM construction are actually free, and we will use the above definition in this paper. The construction of projective modules which are not free is an interesting direction for future research.

3.2 Construction of complex quantum instantons

We will use a variation of the celebrated ADHM construction of instantons [1] to construct complex quantum instantons from \mathbb{C} -stable solutions of the complex ADHM equations (6-8).

To begin, let $(B_{kl}, i_k, j_k) \in \vec{B}$ be a complex ADHM datum, and consider the maps:

$$V \otimes \mathfrak{M}_q^I \xrightarrow[\alpha_2]{\alpha_1} \tilde{W} \otimes \mathfrak{M}_q^I \xrightarrow[\beta_2]{\beta_1} V \otimes \mathfrak{M}_q^I$$

defined as follows:

$$\alpha_1 = \begin{pmatrix} B_{11} \otimes \mathbf{1} - \mathbf{1} \otimes x_{11'} \\ B_{12} \otimes \mathbf{1} - \mathbf{1} \otimes x_{12'} \\ j_1 \otimes \mathbf{1} \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} B_{21} \otimes \mathbf{1} - \mathbf{1} \otimes x_{21'} \\ B_{22} \otimes \mathbf{1} - \mathbf{1} \otimes x_{22'} \\ j_2 \otimes \mathbf{1} \end{pmatrix}$$

$$\beta_1 = \begin{pmatrix} -B_{12} \otimes \mathbf{1} + \mathbf{1} \otimes x_{12'} & B_{11} \otimes \mathbf{1} - \mathbf{1} \otimes x_{11'} & i_1 \otimes \mathbf{1} \end{pmatrix}$$

$$\beta_2 = \begin{pmatrix} -B_{22} \otimes \mathbf{1} + \mathbf{1} \otimes x_{22'} & B_{21} \otimes \mathbf{1} - \mathbf{1} \otimes x_{21'} & i_2 \otimes \mathbf{1} \end{pmatrix}$$

Proposition 23. (B_{kl}, i_k, j_k) satisfies the complex ADHM equations (6-8) if and only if the following identities hold:

$$\beta_1 \alpha_1 = \beta_2 \alpha_2 = 0 \tag{45}$$

$$\beta_2 \alpha_1 + \beta_1 \alpha_2 = 0 \tag{46}$$

Proof. It is easy to check that:

$$\beta_1 \alpha_1 = ([B_{11}, B_{12}] + i_1 j_1) \otimes \mathbf{1} + \mathbf{1} \otimes [x_{11'}, x_{12'}]$$

$$\beta_2 \alpha_2 = ([B_{21}, B_{22}] + i_2 j_2) \otimes \mathbf{1} + \mathbf{1} \otimes [x_{21'}, x_{22'}]$$

$$\beta_2 \alpha_1 + \beta_1 \alpha_2 = ([B_{11}, B_{22}] + [B_{21}, B_{12}] + i_1 j_2 + i_2 j_1) \otimes \mathbf{1} + \mathbf{1} \otimes ([x_{11'}, x_{22'}] + [x_{21'}, x_{12'}])$$

so that the statement follows easily from the commutation relations (31) and (32). \square

For points $P = [p_1 : p_2]$ and $Q = [q_1 : q_2]$ in \mathbb{P}^1 , we consider maps:

$$\beta_P = p_1 \beta_1 + p_2 \beta_2 : \tilde{W} \otimes \mathfrak{M}_q^I \rightarrow V \otimes \mathfrak{M}_q^I$$

$$\alpha_Q = q_1 \alpha_1 + q_2 \alpha_2 : V \otimes \mathfrak{M}_q^I \rightarrow \tilde{W} \otimes \mathfrak{M}_q^I$$

It is easy to see that if \vec{B} satisfies (45-46), then:

$$\beta_P \alpha_Q = (p_1 q_2 - p_2 q_1) \beta_1 \alpha_2 \tag{47}$$

Proposition 24. Assume that \vec{B} satisfies the complex ADHM equations (45-46).

1. $\beta_P \alpha_Q$ is injective for all $P \neq Q \in \mathbb{P}^1$.
2. β_P is surjective if and only if \vec{B}_P is stable.

In particular, β_P is surjective $\forall P \in \mathbb{P}^1$ if and only if \vec{B} is \mathbb{C} -stable.

It also follows easily that α_Q is injective $\forall Q \in \mathbb{P}^1$, so that $\text{Im } \alpha_Q$ is a free submodule of $\tilde{W} \otimes \mathfrak{M}_q^I$, of rank c . Furthermore, $\ker \beta_P \cap \text{Im } \alpha_Q = \{0\}$ for all $P \neq Q \in \mathbb{P}^1$.

Proof. For the first statement, it is enough to show that $\beta_1 \alpha_2$ is injective, by (47). Note that any $\nu \in V \otimes \mathfrak{M}_q^I$ can be presented as a sum $\nu = \nu_d + \nu_{d-1} + \dots + \nu_0$ where

$$\nu_d = \sum_k v_k \otimes f_k \quad , \quad v_k \in V, f_k \in \mathfrak{M}_q^I \text{ has degree } d \quad ,$$

so that $\beta_1 \alpha_2(\nu) = (x_{11'} x_{22'} - x_{12'} x_{21'}) \nu_d + (\text{terms of lower degree})$.

We argue that the endomorphism of \mathfrak{M}_q^I given by the multiplication by $(x_{11'} x_{22'} - x_{12'} x_{21'})$ is injective. In fact, we can present an element $f \in \mathfrak{M}_q^I$ in the basis of monomials (38). Let us choose the lexicographic order of such basis, i.e. $(n_{11'}, n_{12'}, n_{21'}, n_{22'}) > (m_{11'}, m_{12'}, m_{21'}, m_{22'})$ if $n_{11'} > m_{11'}$, or $n_{11'} = m_{11'}$ and $n_{12'} > m_{12'}$, or $n_{11'} = m_{11'}$ and $n_{12'} = m_{12'}$ and $n_{21'} > m_{21'}$, or $n_{11'} = m_{11'}$ and $n_{12'} = m_{12'}$ and $n_{21'} = m_{21'}$ and $n_{22'} > m_{22'}$. Then the commutation relations (31-32) imply that, in this basis, the multiplication by $x_{ij'}$ increases the exponent $n_{ij'}$ by 1 and multiplies its coefficient by a power of q . Thus the multiplication by $(x_{11'} x_{22'} - x_{12'} x_{21'})$ of a polynomial with a nonzero leading monomial of the form (38) will yield a polynomial with a nonzero leading monomial of the form $x_{11'}^{n_{11'}+1} x_{12'}^{n_{12'}} x_{21'}^{n_{21'}} x_{22'}^{n_{22'}+1}$. Thus indeed, multiplication by $(x_{11'} x_{22'} - x_{12'} x_{21'})$ induces an injective endomorphism of \mathfrak{M}_q^I , as desired.

Now, if $\beta_1 \alpha_2(\nu) = 0$, then $(x_{11'} x_{22'} - x_{12'} x_{21'}) \nu_d = 0$, which in turn implies that $\nu_d = 0$; by induction on d , we conclude that $\nu = 0$.

For the second statement, consider the following polynomial algebras for each $[p_1 : p_2] \in \mathbb{P}^1$:

$$\chi_P = \mathbb{C}[p_1x_{11'}+p_2x_{21'}, p_1x_{12'}+p_2x_{22'}] \text{ and } \chi_{\overline{P}} = \mathbb{C}[p_2x_{11'}-p_1x_{21'}, p_2x_{12'}-p_1x_{22'}]$$

as commutative subalgebras of \mathfrak{M}_q^I ; clearly, $\mathfrak{M}_q^I = \chi_P \otimes \chi_{\overline{P}} / \sim$, where by " \sim " we understand the commutation relations between the generators of χ_P and those of $\chi_{\overline{P}}$, which can be deduced from (31-32).

As in [4, Proposition 10], we see that, for each $P \in \mathbb{P}^1$, β_P restricts to a map $\beta_P|_{\chi_P} : \tilde{W} \otimes \chi_P \rightarrow V \otimes \chi_P$, and that $\beta_P = \beta_P|_{\chi_P} \otimes \mathbf{1}_{\chi_{\overline{P}}}$. To complete the proof, recall from [10, Lemma 2.7] that $\beta_P|_{\chi_P}$ is surjective if and only if \vec{B}_P is stable. \square

Now we consider the maps $\vec{\alpha} : (V \oplus V) \otimes \mathfrak{M}_q^I \rightarrow \tilde{W} \otimes \mathfrak{M}_q^I$ and $\vec{\beta} : \tilde{W} \otimes \mathfrak{M}_q^I \rightarrow (V \oplus V) \otimes \mathfrak{M}_q^I$ given by:

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \quad \text{and} \quad \vec{\beta} = \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix}$$

If \vec{B} satisfies the complex ADHM equations, the identities in Proposition 23 imply that $\Xi = \vec{\beta}\vec{\alpha} = \beta_1\alpha_2\mathbf{1}_{\mathbb{C}^2}$. It follows that Ξ is injective; in particular, $\vec{\alpha}$ is injective and $\text{Im } \vec{\alpha} = \text{Im } \alpha_1 \oplus \text{Im } \alpha_2$ is a free submodule of $\tilde{W} \otimes \mathfrak{M}_q^I$, of rank $2c$. Moreover, $\ker \vec{\beta} \cap \text{Im } \vec{\alpha} = \{0\}$.

Proposition 25. *If \vec{B} is a \mathbb{C} -regular solution of the complex ADHM equations, then the map $\beta_1\alpha_2 : V \otimes \mathfrak{M}_q^I \rightarrow V \otimes \mathfrak{M}_q^I$ is an isomorphism; if $\beta_1\alpha_2$ is an isomorphism, then \vec{B} is \mathbb{C} -stable.*

Proof. We already know that $\beta_1\alpha_2$ is injective. If \vec{B} is \mathbb{C} -regular, then $\beta_1\alpha_2$ is also surjective in the classical case $q = 1$. Since $\dim \text{coker } \beta_1\alpha_2$ cannot jump for generic value of the parameter q , we obtain the first statement.

Now if $\beta_1\alpha_2$ is an isomorphism then, according to (47), β_P is surjective for all $P \in \mathbb{P}^1$; thus \vec{B} is a \mathbb{C} -stable by Proposition 24. \square

In particular, if \vec{B} is a \mathbb{C} -regular solution of (6-8), then $\Xi = \vec{\beta}\vec{\alpha}$ is an isomorphism, and we define the map:

$$P : \tilde{W} \otimes \mathfrak{M}_q^I \rightarrow \tilde{W} \otimes \mathfrak{M}_q^I$$

$$P = \mathbf{1}_{\tilde{W} \otimes \mathfrak{M}_q^I} - \vec{\alpha}(\Xi)^{-1}\vec{\beta}$$

and notice that $P^2 = P$, i.e. P is a projection. Note that:

$$\text{Im}(P) = \ker \vec{\beta} = \ker \beta_1 \cup \ker \beta_2 .$$

The right \mathfrak{M}_q^I -module $E = \text{Im}(P)$ is finitely generated and stably-free, since $\ker P = \text{Im } \vec{\alpha}$ is free ($\vec{\alpha}$ is injective) and $E \oplus \ker P = \tilde{W} \otimes \mathfrak{M}_q^I$. Furthermore, \mathfrak{M}_q^I is Noetherian and every stably-free module over a noetherian ring is free. Thus we conclude that E is a free \mathfrak{M}_q^I -module.

A connection ∇ on E can be easily defined via the projection formula, as usual:

$$\nabla : E \xrightarrow{\iota} \tilde{W} \otimes \mathfrak{M}_q^I \xrightarrow{1 \otimes d} \tilde{W} \otimes \Omega_{\mathfrak{M}_q^I}^1 \xrightarrow{P \otimes 1} E \otimes_{\mathfrak{M}_q^I} \Omega_{\mathfrak{M}_q^I}^1$$

where ι is the natural inclusion.

We recall that ∇ can be associated to a *connection form* $A \in \text{End}(W) \otimes$, so that $\nabla = \nabla_A = d + A$, see [4, page 485]. Furthermore, there is a gauge in which $A = \Psi^{-1}d\Psi = -(d\Psi^{-1})\Psi$ for an isomorphism $\Psi : W \otimes \mathfrak{M}_q^I \rightarrow E$ [4, page 494].

Proposition 26. ∇ is anti-self-dual.

The argument here is again very similar to the one in [4, Proposition 14]; we repeat it here for the sake of completeness.

Proof. Note that $F_\nabla = \nabla\nabla = PdPd$; therefore we have:

$$\begin{aligned} F_\nabla &= P \left(d(\mathbf{1}_{\tilde{W} \otimes \mathfrak{M}_q^I} - \vec{\alpha}\Xi^{-1}\vec{\beta})d \right) = P \left(d\vec{\alpha}\Xi^{-1}(d\vec{\beta}) \right) = \\ &= P \left((d\vec{\alpha})\Xi^{-1}(d\vec{\beta}) + \vec{\alpha}d(\Xi^{-1}(d\vec{\beta})) \right) = \\ &= P \left((d\vec{\alpha})\Xi^{-1}(d\vec{\beta}) \right) \end{aligned}$$

for $P\vec{\alpha}d(\Xi^{-1}(d\mathcal{D}_1)) = 0$. Since $\Xi^{-1} = (\beta_1\alpha_2)^{-1}\mathbf{1}_{\mathbb{C}^2}$, we conclude that F_{∇} is proportional to $d\vec{\alpha} \wedge d\vec{\beta}$, as a 2-form.

It is then a straightforward calculation to show that each entry of $d\vec{\alpha} \wedge d\vec{\beta}$ belongs to $\Omega_{\mathfrak{M}_q^I}^{2,-}$; indeed:

$$\begin{aligned} d\vec{\alpha} \wedge d\vec{\beta} &= \begin{pmatrix} -dx_{11'} & -dx_{21'} \\ -dx_{12'} & -dx_{22'} \\ 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} dx_{22'} & -dx_{21'} & 0 \\ -dx_{12'} & dx_{11'} & 0 \end{pmatrix} = \\ &= \begin{pmatrix} -dx_{11'}dx_{22'} + dx_{21'}dx_{12'} & dx_{11'}dx_{21'} - dx_{21'}dx_{11'} & 0 \\ -dx_{12'}dx_{22'} + dx_{22'}dx_{12'} & dx_{12'}dx_{21'} - dx_{22'}dx_{11'} & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Applying the commutation relations (41), we obtain:

$$d\vec{\alpha} \wedge d\vec{\beta} = \begin{pmatrix} -(dx_{11'}dx_{22'} + dx_{12'}dx_{21'}) & 2dx_{11'}dx_{21'} & 0 \\ -2dx_{12'}dx_{22'} & dx_{11'}dx_{22'} + dx_{12'}dx_{21'} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Comparison with (43) completes the proof. \square

The same procedure can be used to construct complex quantum instantons on \mathfrak{M}_q^J ; consider the maps:

$$V \otimes \mathfrak{M}_q^J \xrightarrow[\alpha_2]{\alpha_1} \tilde{W} \otimes \mathfrak{M}_q^J \xrightarrow[\beta_2]{\beta_1} V \otimes \mathfrak{M}_q^J$$

defined as follows:

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} B_{11} \otimes \mathbf{1} - y_{22'} \\ B_{12} \otimes \mathbf{1} + \mathbf{1} \otimes y_{12'} \\ j_1 \otimes \mathbf{1} \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} B_{21} \otimes \mathbf{1} + y_{21'} \\ B_{22} \otimes \mathbf{1} - \mathbf{1} \otimes y_{11'} \\ j_2 \otimes \mathbf{1} \end{pmatrix} \\ \beta_1 &= \begin{pmatrix} -B_{12} \otimes \mathbf{1} - \mathbf{1} \otimes y_{12'} & B_{11} \otimes \mathbf{1} - \mathbf{1} \otimes y_{22'} & i_1 \otimes \mathbf{1} \end{pmatrix} \\ \beta_2 &= \begin{pmatrix} -B_{22} \otimes \mathbf{1} + \mathbf{1} \otimes y_{11'} & B_{21} \otimes \mathbf{1} + \mathbf{1} \otimes y_{21'} & i_2 \otimes \mathbf{1} \end{pmatrix} \end{aligned}$$

Again, we set $\vec{\alpha} : (V \oplus V) \otimes \mathfrak{M}_q^I \rightarrow \tilde{W} \otimes \mathfrak{M}_q^J$ and $\vec{\beta} : \tilde{W} \otimes \mathfrak{M}_q^I \rightarrow (V \oplus V) \otimes \mathfrak{M}_q^J$ given by:

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \quad \text{and} \quad \vec{\beta} = \begin{pmatrix} -\beta_2 \\ \beta_1 \end{pmatrix}.$$

It follows that if \vec{B} is \mathbb{C} -regular, then $\Xi = \vec{\beta}\vec{\alpha} = \beta_1\alpha_2\mathbf{1}_{\mathbb{C}^2}$ is an isomorphism so that $E = \ker \vec{\beta}$ is a finitely generated free right \mathfrak{M}_q^J -module; an anti-self-dual connection is again produced via the projection formula.

The consistency map Γ is obtained by restricting the obvious map $\tilde{W} \otimes \mathfrak{M}_q^I[\det(x)^{-1}] \rightarrow \tilde{W} \otimes \mathfrak{M}_q^J[\det(y)^{-1}]$, see (37); further details can be found in [4].

Finally, it is easy to see that $GL(V)$ -equivalent complex ADHM data will lead to gauge equivalent quantum instantons (see [4]). Our next result is the following:

Theorem 27. *There is a well-defined map from the set of equivalence classes of \mathbb{C} -regular solutions of the complex ADHM equations to the moduli space of gauge equivalence classes of complex quantum instantons on $\mathfrak{M}_{p,q}$.*

By Proposition (25), \mathbb{C} -stability is a necessary condition for the ADHM construction of instanton, possibly also sufficient. In that case, the domain of the map in the theorem would be enlarged to the set of equivalence classes of \mathbb{C} -stable solutions of the complex ADHM equations.

3.3 Quantum Laplacian and admissibility

In this final section we will relate the admissibility condition for sheaves on \mathbb{P}^3 and solutions of the Laplace equation in the quantum Minkowski space-time \mathfrak{M}_q^I , thus extending the classical Penrose correspondence [16]. A q -deformation of the Penrose transform with the quantum, rather than classical, twistor space has been studied in [13] (see also the references in [13]).

We begin by constructing the quantum counterpart of the Laplace equation. Let us define the quantum partial derivatives $\partial_{rs'}$ by the relation:

$$df = (\partial_{11'}f) dx_{11'} + (\partial_{12'}f) dx_{12'} + (\partial_{21'}f) dx_{21'} + (\partial_{22'}f) dx_{22'} \quad (48)$$

where $f \in \mathfrak{M}_q^1$. Then the property $d^2 = 0$ implies the following commutation relations:

$$\partial_{11'}\partial_{21'} = \partial_{21'}\partial_{11'}, \quad \partial_{12'}\partial_{22'} = \partial_{22'}\partial_{12'}, \quad (49)$$

$$[\partial_{11'}, \partial_{22'}] + [\partial_{12'}, \partial_{21'}] = 0 \quad (50)$$

$$\begin{aligned} \partial_{11'}\partial_{12'} &= q^{-2}\partial_{12'}\partial_{11'}, & \partial_{21'}\partial_{22'} &= q^{-2}\partial_{22'}\partial_{21'}, \\ \partial_{12'}\partial_{21'} &= q^2\partial_{21'}\partial_{12'} \end{aligned} \quad (51)$$

We define the quantum Laplacian by:

$$\square_x = \partial_{11'}\partial_{22'} - \partial_{21'}\partial_{12'} = \partial_{22'}\partial_{11'} - \partial_{12'}\partial_{21'}, \quad (52)$$

Alternatively, note that the quantum Laplacian can also be expressed via the Hodge star involution, just as in the classical case:

$$\square_x = *d * d, \quad (53)$$

where the Hodge star on 0- and 1-forms is defined by the following (cf. [4, Section 2.1]):

$$*1 = q^{-1}dx_{11'} \wedge dx_{12'} \wedge dx_{21'} \wedge dx_{22'} = qdx_{22'} \wedge dx_{21'} \wedge dx_{12'} \wedge dx_{11'} \quad (54)$$

$$\begin{aligned} *dx_{11'} &= -\frac{1}{[2]}dx_{11'} \wedge dx_{12'} \wedge dx_{21'} \\ *dx_{12'} &= -\frac{1}{[2]}dx_{12'} \wedge dx_{22'} \wedge dx_{11'} \\ *dx_{21'} &= -\frac{1}{[2]}dx_{21'} \wedge dx_{11'} \wedge dx_{22'} \\ *dx_{22'} &= -\frac{1}{[2]}dx_{22'} \wedge dx_{21'} \wedge dx_{12'} \end{aligned} \quad (55)$$

where $[n]$ for $n \in \mathbb{Z}$, denotes the quantum integer $(q^n - q^{-n})/(q - q^{-1})$.

We will now present a basis of solutions of the quantum Laplace equation

$$\square_x f = 0, \quad f \in \mathfrak{M}_q^1 \quad (56)$$

in integral form, exactly as in the classical case [16].

Proposition 28. *The following elements:*

$$X_{mn}^l = \frac{1}{2\pi i} \oint (x_{11'}s + x_{21'})^{l-m} (x_{12'}s + x_{22'})^{l+m} s^{n-l-1} ds, \quad (57)$$

$$-l \leq n, m \leq l, \quad l \in \frac{1}{2}\mathbb{Z}_+, \quad n, m \equiv l \pmod{1}$$

form a basis of solutions of the quantum Laplace equation (56), where the integration variable s commutes with the generators $x_{kl'}$.

Proof. First we will show that the above elements satisfy the quantum Laplace equation. We note that since

$$(x_{11'}s + x_{21'})(x_{12'}s + x_{22'}) = (x_{12'}s + x_{22'})(x_{11'}s + x_{21'})$$

the expression (57) does not depend on the order of the factors. Also, the commutation relations (40) imply in particular:

$$(dx_{11'}s + dx_{21'})(x_{11'}s + x_{21'}) = p^2(x_{11'}s + x_{21'})(dx_{11'}s + dx_{21'}) \quad (58)$$

$$(dx_{12'}s + dx_{22'})(x_{12'}s + x_{22'}) = p^2(x_{12'}s + x_{22'})(dx_{12'}s + dx_{22'}) \quad (59)$$

for $p = q^{\pm 1}$ and also

$$(dx_{11'}s + dx_{21'})(x_{12'}s + x_{22'}) = p^2(x_{12'}s + x_{22'})(dx_{11'}s + dx_{21'}) \quad (60)$$

for $p = q$ only and

$$(dx_{12'}s + dx_{22'})(x_{11'}s + x_{21'}) = p^2(x_{11'}s + x_{21'})(dx_{12'}s + dx_{22'}) \quad (61)$$

for $p = q^{-1}$ only. Therefore, in the computation of the differential dX_{mn}^l , the order of factors has to be chosen accordingly. For $p = q$ we obtain:

$$\begin{aligned} dX_{mn}^l &= \frac{1}{2\pi i} \oint d \left((x_{11'}s + x_{21'})^{l-m} (x_{12'}s + x_{22'})^{l+m} \right) s^{n-l-1} ds = \\ &\left(\sum_{k=0}^{l+m-1} q^{2k} \right) \oint (x_{11'}s + x_{21'})^{l-m} (x_{12'}s + x_{22'})^{l+m-1} (dx_{12'}s + dx_{22'}) s^{n-l-1} ds + \\ &+ \left(\sum_{k=l+m}^{2l-1} q^{2k} \right) \oint (x_{11'}s + x_{21'})^{l-m-1} (x_{12'}s + x_{22'})^{l+m} (dx_{11'}s + dx_{21'}) s^{n-l-1} ds \end{aligned}$$

Similarly, for $p = q^{-1}$ we obtain:

$$\begin{aligned}
dX_{mn}^l &= \frac{1}{2\pi i} \oint d((x_{12'}s + x_{22'})^{l+m}(x_{11'}s + x_{21'})^{l-m}) s^{n-l-1} ds = \\
&= \left(\sum_{k=1}^{l-m-1} q^{-2k} \right) \oint (x_{12'}s + x_{22'})^{l+m}(x_{11'}s + x_{21'})^{l-m-1} (dx_{11'}s + dx_{21'}) s^{n-l-1} ds + \\
&+ \left(\sum_{k=l-m}^{2l-1} q^{-2k} \right) \oint (x_{12'}s + x_{22'})^{l+m-1}(x_{11'}s + x_{21'})^{l-m} (dx_{12'}s + dx_{22'}) s^{n-l-1} ds
\end{aligned}$$

These yield the explicit expressions for the quantum partial derivatives, which can be written uniformly for $p = q^{\pm 1}$ as follows:

$$\begin{aligned}
\partial_{11'} X_{mn}^l &= p^{2l-1} q^{m+l} [l-m] X_{(n+\frac{1}{2})(m+\frac{1}{2})}^{l-\frac{1}{2}} \\
\partial_{12'} X_{mn}^l &= p^{2l-1} q^{m-l} [l+m] X_{(n+\frac{1}{2})(m-\frac{1}{2})}^{l-\frac{1}{2}} \\
\partial_{21'} X_{mn}^l &= p^{2l-1} q^{m+l} [l-m] X_{(n-\frac{1}{2})(m+\frac{1}{2})}^{l-\frac{1}{2}} \\
\partial_{22'} X_{mn}^l &= p^{2l-1} q^{m-l} [l+m] X_{(n-\frac{1}{2})(m-\frac{1}{2})}^{l-\frac{1}{2}}
\end{aligned} \tag{62}$$

The relations (62) immediately imply:

$$\square_x X_{nm}^l = 0 \tag{63}$$

Finally, we need to argue that any solution of the quantum Laplace equations (56) is a (complex) linear combination of the ones above. In fact, as in the classical case, the elements:

$$(\det(x))^k X_{nm}^l, \quad k \in \mathbb{Z}_+, \quad l \in \frac{1}{2}\mathbb{Z}_+, \quad -l \leq n, m \leq l \tag{64}$$

$$\text{where } \det(x) = x_{11'}x_{22'} - x_{12'}x_{21'} = x_{22'}x_{11'} - x_{21'}x_{12'}$$

form a basis of \mathfrak{M}_q^l , since they are linearly independent (even for $q = 1$), and the number of elements of fixed degree in (64) is the same as the number of ordered monomials (38) on the four variables $x_{rs'}$, which also compose a basis of \mathfrak{M}_q^l .

On the other hand, for $q = 1$, the elements (64) with $k = 0$ form a basis of solutions of the Laplace equations, known as harmonic polynomials, and this space cannot increase for generic or formal parameter q . \square

In fact, one can prove an explicit quantum analog of the spectral decomposition of the Laplace operator, which contains, as a special case, the statement of the Proposition 34.

Proposition 29. *The basis (64) consists of eigenfunctions of the operator*

$$\tilde{\square}_x = \det(x) \cdot \square_x \quad (65)$$

with eigenvalues $p^{2k+2l-3}[k][k+2l+1]$.

The proof requires an elementary quantum calculus, which will be given in Appendix C.

Now we would like to reinterpret Proposition 28 in terms of the sheaf cohomology of $\mathbb{P}^I = \mathbb{P}^3 \setminus \ell_\infty$. We will consider a covering of \mathbb{P}^I by its two simply connected patches:

$$\mathbb{P}_{(1)}^I = \{[x : y : z : w] \in \mathbb{P}^3 \mid z \neq 0\}$$

$$\mathbb{P}_{(2)}^I = \{[x : y : z : w] \in \mathbb{P}^3 \mid w \neq 0\}$$

Then the elements of $H^1(\mathbb{P}^I, \mathcal{O}_{\mathbb{P}^I}(-2))$ can be represented by transition functions which are rational in x, y, z, w of degree -2 , with singularities along the hyperplanes $\{z = 0\}$ and $\{w = 0\}$. A natural basis of such functions is given by the Laurent monomials

$$\frac{x^{l-m} y^{l+m}}{z^{l-n+1} w^{l+n+1}}, \quad l \in \frac{1}{2}\mathbb{Z}_+, \quad -l \leq n, m \leq l. \quad (66)$$

The quantum Penrose transform assigns to an element of this basis the quantum harmonic polynomials X_{nm}^l via the formula (57). Thus we can restate Proposition 28 as follows.

Proposition 30. *There is an isomorphism*

$$H^1(\mathbb{P}^I, \mathcal{O}_{\mathbb{P}^I}(-2)) \xrightarrow{\cong} \ker \square_x$$

given by the integral formula

$$f(x, y, z, w) \mapsto \frac{1}{2\pi i} \oint f(x_{11'}s + x_{21'}, x_{12'}s + x_{22'}, s, 1) ds$$

where f represents a cocycle in $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$, that is, a linear combination of the terms in (66).

Next we will extend this isomorphisms to more general vector bundles over \mathbb{P}^1 . More precisely, if a vector bundle $\mathcal{E}^I \rightarrow \mathbb{P}^1$ is admissible (i.e. it is the restriction of an admissible vector bundle $\mathcal{E} \rightarrow \mathbb{P}^3$ to \mathbb{P}^1), then a class in $H^1(\mathbb{P}^1, \mathcal{E}^I(-2))$ can be represented by a cocycle $\vec{f} = (f_1, \dots, f_r)$, where r is the rank of \mathcal{E}^I , and each f_k is a linear combination of the terms in (66).

On the other hand, given a complex quantum instanton (E, ∇_A) on \mathfrak{M}_q^I we define the quantum coupled Laplacian by generalizing (53):

$$\square_x^A = * \nabla_A * \nabla_A \quad . \quad (67)$$

To any given admissible vector bundle $\mathcal{E}^I \rightarrow \mathbb{P}^1$ we can associate a regular ADHM data \vec{B} , which can then be used to construct a complex quantum instanton (E, ∇_A) on \mathfrak{M}_q^I . For \mathcal{E}^I and (E, ∇_A) related as above, our next theorem generalizes the classical Penrose correspondence between $H^1(\mathbb{P}^1, \mathcal{E}^I(-2))$ and the set of solutions of the coupled quantum Laplace equation.

Theorem 31. *There is an isomorphism*

$$H^1(\mathbb{P}^1, \mathcal{E}^I(-2)) \xrightarrow{\simeq} \ker \square_x^A$$

given by the integral formula

$$f(x, y, z, w) \mapsto \varphi = \frac{1}{2\pi i} \oint \Psi^{-1} \vec{f}(x_{11'}s + x_{21'}, x_{12'}s + x_{22'}, s, 1) ds$$

Proof. First recall that there is a gauge such that $\nabla_A = d - (d\Psi^{-1})\Psi$. We then have:

$$\begin{aligned} \nabla_A \varphi &= \frac{1}{2\pi i} \oint (d - (d\Psi^{-1})\Psi) \Psi^{-1} \vec{f}(x_{11'}s + x_{21'}, x_{12'}s + x_{22'}, s, 1) ds = \\ &= \frac{1}{2\pi i} \oint \left((d\Psi^{-1})\vec{f} + \Psi^{-1}d\vec{f} - (d\Psi^{-1})\vec{f} \right) ds = \frac{1}{2\pi i} \oint \left(\Psi^{-1}d\vec{f} \right) ds \end{aligned}$$

By the same token, it follows that:

$$\begin{aligned}\square_x^A \varphi &= * \frac{1}{2\pi i} \oint (d - (d\Psi^{-1})\Psi)\Psi^{-1}(*d\vec{f}) \, ds = \\ &= * \frac{1}{2\pi i} \oint \Psi^{-1}(d * d\vec{f}) \, ds = \frac{1}{2\pi i} \oint \Psi^{-1}(\square_x \vec{f}) \, ds = 0\end{aligned}$$

since $\square_x \vec{f} = 0$ by Proposition 30. \square

We now consider global solutions of the quantum Laplace equation on $\mathfrak{M}_{p,q}$. To do that, we have to check the consistency of the solutions in \mathfrak{M}_q^I and \mathfrak{M}_q^J .

We introduce the following elements of \mathfrak{M}_q^J :

$$Y_{nm}^l = \frac{1}{2\pi i} \oint (y_{11'}t + y_{12'})^{l-n} (y_{21'}t + y_{22'})^{l+n} t^{m-l-1} \, dt \quad , \quad (68)$$

$$-l \leq n, m \leq l \quad , \quad l \in \frac{1}{2}\mathbb{Z}_+ \quad , \quad n, m \equiv l \pmod{1} \quad .$$

Then one defines the quantum partial derivatives and the quantum Laplacian \square_y in \mathfrak{M}_q^J as above. The counterparts of Propositions 28 and 29 hold with the eigenvalues of $\tilde{\square}_y$ being equal to $p^{-2k-2l+3}[k][k+2l+1]$ on the basis elements $(\det(y))^k Y_{nm}^l$, where $\det(y)$ was defined in Section 3.1

Including the negative powers of $\det(y)$ in the above basis and the negative powers of $\det(x)$ in the basis (64), we obtain two bases of \mathfrak{M}_q^{IJ} . We can also extend the quantum Laplacians \square_x and \square_y in \mathfrak{M}_q^{IJ} so that Proposition 29 and its counterpart in the y generators hold. The comparison between the two systems of coordinates gives essentially the same result as in the classical $q = 1$ case.

Proposition 32. *In \mathfrak{M}_q^{IJ} , one has:*

- $(\det(x))^k X_{nm}^l$ is proportional to $(\det(y))^{-k-2l} Y_{nm}^l$;
- $\tilde{\square}_x = p^{-8}(\det(y))\tilde{\square}_y(\det(y))^{-1}$.

Proof. The first statement follows from the definitions (57) and (68); for the details, see Appendix C.

The second identity follows from the comparison of the eigenvalues of the two operators in the proportional bases $(\det(x))^k X_{nm}^l$ and $(\det(y))^{-k-2l} Y_{nm}^l$, as computed in Proposition 29. It can also be verified directly by relating the partial derivatives with respect to $x_{kl'}$ and $y_{kl'}$. \square

Thus, as in the classical case, we conclude that there are no consistent solutions to the scalar quantum Laplace equation in the (compactified) quantum Minkowski space-time $\mathfrak{M}_{p,q}$. With Proposition 30 in mind, this non-existence statement corresponds to the fact that $H^1(\mathbb{P}^3, \mathcal{O}(-2)) = 0$.

More generally, let \mathcal{E} be an admissible vector bundle over \mathbb{P}^3 , to which we can associate, via the intermediate \mathbb{C} -regular ADHM datum, a complex quantum instanton $(E_I, \nabla_I; E_J, \nabla_J)$ over $\mathfrak{M}_{p,q}$. Taking the non-existence of consistent solutions to the scalar quantum Laplace equation to the context of Theorem 31, we conclude that there are no consistent solutions to the coupled quantum Laplace equation. This in turn corresponds to the vanishing of $H^1(\mathbb{P}^3, \mathcal{E}(-2))$.

In a future paper, we plan to establish the reverse correspondence: given a complex quantum instanton over $\mathfrak{M}_{p,q}$, we will associate directly an admissible vector bundle over \mathbb{P}^3 . This will generalize the celebrated Penrose-Ward correspondence.

A Moduli space of stable ADHM data

Here we recall the proof that $\mathcal{M}(r, c)$ is a smooth complex manifold of dimension $2rc$; some of the arguments were relevant in Section 1.2, for the proof of Theorem 8. Our arguments are inspired by [6, 10, 15].

Let V and W be complex vector spaces, with dimensions c and r , respectively, and set $\tilde{W} = V \oplus V \oplus W$. Define also:

$$\mathbf{B} = \text{Hom}(V, V) \oplus \text{Hom}(V, V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$$

A point $(B_k, i, j) \in \mathbf{B}$ ($k = 1, 2$) is called a *ADHM datum*. As mentioned above, the groups $GL(V)$ and $GL(W)$ act on \mathbf{B} in the following way:

$$g \cdot (B_k, i, j) = (gB_k g^{-1}, gi, jg^{-1}) , \quad g \in GL(V) \quad (69)$$

$$g \cdot (B_k, i, j) = (B_k, ig^{-1}, gj) , \quad g \in GL(W) \quad (70)$$

Theorem 33. $\mathcal{M}(r, c)$ is a smooth, complex manifold of dimension $2rc$.

Indeed, it is known that $\mathcal{M}(r, c)$ is non-empty for all $r, c \geq 1$. Furthermore, it can be shown that $\mathcal{M}(r, c)$ is a simply-connected quasi-projective algebraic variety [15] and that it admits a complete hyperkähler metric [10]. The strategy of the proof goes as follows; considering the map:

$$\begin{aligned} \mu : \mathbf{B}^{\text{st}} &\rightarrow \text{Hom}(V, V) \\ \mu(B_1, B_2, i, j) &= [B_1, B_2] + ij \end{aligned}$$

where \mathbf{B}^{st} is the open subset of stable ADHM data. We first show that $GL(V)$ acts freely in \mathbf{B}^{st} , and that the action has a closed graph; we then show that $\mu^{-1}(0)$ is indeed a complex manifold of dimension $2rc + c^2$; the desired result follows from general theory (see for instance the *closed graph lemma* in [6] and the references therein).

Proposition 34. (B_1, B_2, i, j) is stable if and only if:

1. (B_1, B_2, i, j) is not fixed of the $GL(V)$ action;
2. if $X \in \text{Hom}(V, V)$ satisfies $[B_1, X] = [B_2, X] = Xi = 0$, then $X = 0$.

Proof. Suppose that (B_1, B_2, i, j) is fixed by some $g \neq \mathbf{1}_V \in GL(V)$, so that, $gB_k g^{-1} = B_k$ ($k = 1, 2$) and $gi = i$. The former implies that $\ker(g - \mathbf{1}_V)$ is B_k invariant, while the latter implies that $\ker(g - \mathbf{1}_V) \subset \text{Im} i$, thus contradicting stability.

For the second statement, $Xi = 0$ implies that $i(W) \subset \ker X$, while $[B_k, X] = 0$ implies that $\ker X$ is B_k -invariant. Stability then implies that $X = 0$. \square

Proposition 35. *The action (69) has a closed graph, i.e. the set*

$$\Gamma = \{(X, Y) \in \mathbf{B}^{\text{st}} \times \mathbf{B}^{\text{st}} \mid Y = g \cdot X \text{ for some } g \in GL(V)\}$$

is closed in $\mathbf{B}^{\text{st}} \times \mathbf{B}^{\text{st}}$. In other words, $GL(V)$ acts properly in \mathbf{B}^{st} .

Proof. Let $\{X_k\}$ be a sequence in \mathbf{B}^{st} , while $\{g_k\}$ denotes a sequence in $GL(V)$; assuming that:

$$\lim_{k \rightarrow \infty} X_k = X_\infty \quad \text{and} \quad \lim_{k \rightarrow \infty} g_k \cdot X_k = Y_\infty ,$$

we must show that $Y_\infty = g_\infty \cdot X_\infty$ for some $g_\infty \in GL(V)$, or equivalently that the sequence $\{g_k\}$ converges to $g_\infty \in GL(V)$.

Indeed, for any given $X = (B_1, B_2, i, j) \in \mathbf{B}^{\text{st}}$ we consider the map $R(X) : W^{\oplus c^2} \longrightarrow V$ given by $(1 \leq m, n \leq c-1)$:

$$R(X) = i \oplus \cdots \oplus B_1^m B_2^n i \oplus \cdots \oplus B_1^{c-1} B_2^{c-1} i .$$

Note that $gR(X) = R(g \cdot X)$ for any $g \in GL(V)$.

Furthermore, $R(X)$ is surjective if and only if X is stable. Indeed, if $X = (B_1, B_2, i, j)$ is not stable, then there is $v \in V^*$ such that $B_1^* v = \lambda_1 v$, $B_2^* v = \lambda_2 v$ and $i^* v = 0$; hence $R^* v = 0$ so that R is not surjective. Conversely, if R is not surjective, then $S = \text{Im } R$ is a proper subspace of V ; clearly, S is B_1 and B_2 invariant, and $i(W) \subset S$, hence X is not stable.

The sequence of maps $R(X_k)$ converges to $R(X_\infty)$; thus, there is a sequence of maps $T_k \in \text{Hom}(V, W^{\oplus c^2})$ converging to a map $T_\infty \in \text{Hom}(V, W^{\oplus c^2})$ such that:

$$W^{\oplus c^2} = \ker R(X_k) \oplus \text{Im } T_k = \ker R(X_\infty) \oplus \text{Im } T_\infty$$

It then follows that $R(X_k)T_k$ and $R(X_\infty)T_\infty$ are invertible as operators on V .

Now set $g_\infty = R(Y_\infty)T_\infty[R(X_\infty)T_\infty]^{-1} \in GL(V)$. Thus:

$$g_k = g_k[R(X_k)T_k][R(X_k)T_k]^{-1} = [R(g_k X_k)T_k][R(X_k)T_k]^{-1}$$

and g_k converges to g_∞ . □

Proposition 36. $\vec{B} = (B_1, B_2, i, j)$ is stable if and only if the derivative map $D_{\vec{B}}\mu : \mathbf{B} \rightarrow \text{Hom}(V, V)$ is surjective.

This means that 0 is a regular value of the map μ , hence $\mu^{-1}(0)$ is a smooth complex manifold of dimension $2rc + c^2$.

Proof. Taking $(b_1, b_2, c, d) \in \mathbf{B}$, the derivative map is given by:

$$D_{\vec{B}}\mu(b_1, b_2, c, d) = [b_1, B_2] + [B_1, b_2] + id + cj$$

Let $X \in \text{Hom}(V, V)$ be orthogonal to the image of $D_{\vec{B}}\mu$, that is

$$\text{Tr}(D_{\vec{B}}\mu(b_1, b_2, c, d)X^\dagger) = 0, \quad \forall (b_1, b_2, c, d).$$

Then in particular:

$$\text{Tr}([b_1, B_2]X^\dagger) = \text{Tr}(b_1[X^\dagger, B_2]) = 0 \quad \forall b_1$$

$$\text{Tr}([B_1, b_2]X^\dagger) = \text{Tr}([X^\dagger, B_1]b_2) = 0 \quad \forall b_2$$

$$\text{Tr}(idX^\dagger) = \text{Tr}(X^\dagger id) = 0 \quad \forall d$$

Hence $[X^\dagger, B_1] = [X^\dagger, B_1] = X^\dagger i = 0$, so $X = 0$ by Proposition 34. \square

Since $GL(V)$ acts freely and properly on the smooth manifold $\mu^{-1}(0)$, this completes the proof of Theorem 33. To conclude this section, we also remark upon the following statements, in which by *irregular* we mean neither stable nor costable:

Proposition 37. Every solution of (1) and (2) is:

1. stable, if $\xi > 0$;
2. costable, if $\xi < 0$;
3. either regular or irregular, if $\xi = 0$.

Proof. For the first statement, if (B_1, B_2, i, j) is not stable, then by duality on V there is a proper subspace $S^\perp \subset V$ such that $B_k^\dagger(S^\perp) \subset S^\perp$ and $S^\perp \subset \ker i^\dagger$. So restricting (2) to S^\perp and taking the trace, we conclude that

$$\mathrm{Tr}(ii^\dagger|_{S^\perp}) = \xi \cdot \dim S + \mathrm{Tr}(j^\dagger j|_{S^\perp}) > 0$$

which yields a contradiction. The proof of the second statement is similar, while the third statement can be found at [4, Lemma 2]. \square

It is interesting to compare the third part of Proposition 37 with Remark 7: the complex equations are a much more flexible than the real ones.

Proposition 38. [10, p. 24]. *Let $r = 1$. Every stable solution of (1) has $j = 0$. In particular, there are no regular solutions for $r = 1$ and $\xi = 0$.*

B Cohomological calculations

We collect here the proofs for various facts used in Section 2.

Proposition 39. *Let \mathcal{E} be an admissible torsion-free sheaf over \mathbb{P}^3 with $\mathrm{ch}(\mathcal{E}) = r - c[H]^2$ and such that $\mathcal{E}|_{\ell_\infty} = \mathcal{O}_{\ell_\infty}^{\oplus r}$. The following hold:*

1. $h^1(\mathbb{P}^3, \mathcal{E}(-1)) = -\chi(\mathcal{E}(-1)) = c$;
2. $h^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = -\chi(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = c + 2r$;
3. $h^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) = -\chi(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) = c$;
4. $H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1))$.

Proof. Let us first spell out the admissibility condition more precisely:

$$\begin{aligned} H^0(\mathbb{P}^3, \mathcal{E}(k)) &= 0, \quad \forall k \leq -1 & H^1(\mathbb{P}^3, \mathcal{E}(k)) &= 0, \quad \forall k \leq -2 \\ H^2(\mathbb{P}^3, \mathcal{E}(k)) &= 0, \quad \forall k \leq -2 & H^3(\mathbb{P}^3, \mathcal{E}(k)) &= 0, \quad \forall k \leq -3 \end{aligned} \quad (71)$$

The first statement then follows immediately from admissibility, and it only remains for us to show that $\chi(\mathcal{E}(-1)) = -c$. Indeed, note that:

$$\mathrm{ch}(\mathcal{E}(-1)) = r - rh + \left(\frac{r}{2} + c\right)h^2 + \left(-\frac{r}{6} + c\right)h^3$$

$$\mathrm{td}(\mathbb{P}^3) = 1 + 2h + \frac{22}{12}h^2 + h^3$$

Hence it follows:

$$\chi(\mathcal{E}(-1)) = \int_{\mathbb{P}^3} \mathrm{ch}(\mathcal{E}(-1)) \mathrm{td}(\mathbb{P}^3) = -c$$

Now consider the Euler sequence for 1-forms:

$$0 \rightarrow \Omega_{\mathbb{P}^3}^1 \rightarrow \bigoplus_4 \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow 0 \quad (72)$$

from which we conclude that:

$$\begin{aligned} \mathrm{ch}(\Omega_{\mathbb{P}^3}^1) &= 3 - 4h + 2h^2 + \frac{2}{3}h^3 \\ \mathrm{ch}(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) &= 3r + 4rh - (3c - 2r)h^2 - \left(\frac{2r}{3} + 4c\right)h^3 \end{aligned}$$

Using Riemann-Roch again we obtain:

$$\chi(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = \int_{\mathbb{P}^3} \mathrm{ch}(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) \mathrm{td}(\mathbb{P}^3) = -c - 2r$$

Tensoring (72) by \mathcal{E} we obtain the exact sequence:

$$\mathrm{Tor}^1(\mathcal{E}, \mathcal{O}_{\mathbb{P}^3}) \rightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1 \rightarrow \bigoplus_4 \mathcal{E}(-1) \rightarrow \mathcal{E} \rightarrow 0$$

But the first term vanishes because $\mathcal{O}_{\mathbb{P}^3}$ is a locally-free sheaf. Therefore we have:

$$0 \rightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1 \rightarrow \bigoplus_4 \mathcal{E}(-1) \rightarrow \mathcal{E} \rightarrow 0 \quad (73)$$

At the level of cohomology, one obtains:

$$0 \rightarrow H^0(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) \rightarrow \bigoplus_4 H^0(\mathbb{P}^3, \mathcal{E}(-1))$$

and since $H^0(\mathbb{P}^3, \mathcal{E}(-1)) = 0$, it follows that $H^0(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = 0$. Moreover, (73) also implies that:

$$H^2(\mathbb{P}^3, \mathcal{E}) \rightarrow H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) \rightarrow \bigoplus_4 H^3(\mathbb{P}^3, \mathcal{E}(-1))$$

Since the first and third groups vanish by admissibility, we obtain $H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = 0$.

The Euler sequence for 3-forms is given by:

$$0 \rightarrow \Omega_{\mathbb{P}^3}^3 \rightarrow \bigoplus_4 \mathcal{O}_{\mathbb{P}^3}(-3) \rightarrow \Omega_{\mathbb{P}^3}^2 \rightarrow 0 \quad (74)$$

Recalling that $\Omega_{\mathbb{P}^3}^3 = \mathcal{O}_{\mathbb{P}^3}(-4)$ and tensoring (74) by $\mathcal{E}(1)$, we obtain:

$$0 \rightarrow \mathcal{E}(-3) \rightarrow \bigoplus_4 \mathcal{E}(-2) \rightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1) \rightarrow 0 \quad (75)$$

Since $H^2(\mathbb{P}^3, \mathcal{E}(-2)) = 0$ for all p , it follows from the cohomology sequence associated with (75) that:

$$H^p(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) \simeq H^{p+1}(\mathbb{P}^3, \mathcal{E}(-3)) \quad (76)$$

thus $H^p(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) = 0$ for $p = 0, 2, 3$ by admissibility. The sequence (75) can also be used to compute the Chern character of $\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)$; indeed,

$$\begin{aligned} \text{ch}(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) &= 4\text{ch}(\mathcal{E}(-2)) - \text{ch}(\mathcal{E}(-3)) = \\ &= 3r - 5rh + \left(\frac{7r}{2} - 3c\right)h^2 + \left(5c - \frac{5r}{6}\right)h^3 \end{aligned}$$

It then follows that:

$$\chi(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) = \int_{\mathbb{P}^3} \text{ch}(\mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) \text{td}(\mathbb{P}^3) = -c$$

what completes the proof of the third statement.

To complete the proof of the second statement, it only remains for us to show that $H^2(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) = 0$. Tensoring the Euler sequence for 2-forms

$$0 \rightarrow \Omega_{\mathbb{P}^3}^2 \rightarrow \bigoplus_6 \mathcal{O}_{\mathbb{P}^3}(-2) \rightarrow \Omega_{\mathbb{P}^3}^1 \rightarrow 0 \quad (77)$$

by \mathcal{E} we obtain:

$$0 \rightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2 \rightarrow \bigoplus_6 \mathcal{E}(-2) \rightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1 \rightarrow 0 \quad (78)$$

The associated cohomology sequence yields:

$$\bigoplus_6 H^2(\mathbb{P}^3, \mathcal{E}(-2)) \rightarrow H^2(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) \rightarrow H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2) \rightarrow \bigoplus_6 H^3(\mathbb{P}^3, \mathcal{E}(-2))$$

Admissibility implies that the first and last groups vanish, hence $H^2(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^1) \simeq H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2)$. Now tensoring (74) by \mathcal{E} we get:

$$0 \rightarrow \mathcal{E}(-4) \rightarrow \bigoplus_4 \mathcal{E}(-3) \rightarrow \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2 \rightarrow 0 \quad (79)$$

we conclude that $H^3(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2) = 0$ since $H^3(\mathbb{P}^3, \mathcal{E}(-3)) = 0$ by admissibility.

Finally, let \wp be a plane containing ℓ_∞ , so that the restriction $\mathcal{E}|_\wp$ yields a torsion-free sheaf on \wp which is trivial at ℓ_∞ . Consider the sequence:

$$0 \rightarrow \mathcal{E}(-p-1) \rightarrow \mathcal{E}(-p) \rightarrow \mathcal{E}(-p)|_\wp \rightarrow 0$$

Setting $p = -2$, we conclude that $H^1(\mathbb{P}^3, \mathcal{E}(-2)|_\wp) \simeq H^2(\mathbb{P}^3, \mathcal{E}(-3))$, since $H^1(\mathbb{P}^3, \mathcal{E}(-2)) = H^2(\mathbb{P}^3, \mathcal{E}(-2)) = 0$ by admissibility. Then setting $p = -1$, we get that $H^1(\mathbb{P}^3, \mathcal{E}(-1)|_\wp) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1))$ for the same reason. Together with (76), we have obtained the identifications

$$\begin{aligned} H^1(\mathbb{P}^3, \mathcal{E} \otimes \Omega_{\mathbb{P}^3}^2(1)) &\simeq H^2(\mathbb{P}^3, \mathcal{E}(-3)) \simeq \\ &\simeq H^1(\mathbb{P}^3, \mathcal{E}(-2)|_\wp) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1)|_\wp) \simeq H^1(\mathbb{P}^3, \mathcal{E}(-1)) \end{aligned}$$

where the third identification follows from [10, page 20]. This completes the proof of the fourth statement. \square

Proposition 40. *Consider the following exact sequence of sheaves on a regular algebraic variety V of dimension 3:*

$$0 \rightarrow \mathcal{A} \xrightarrow{\mu} \mathcal{B} \rightarrow \mathcal{C} \rightarrow 0 \quad (80)$$

where \mathcal{A} and \mathcal{B} are locally-free. Then:

1. \mathcal{C} is torsion-free if and only if the localized map $\mu_X : \mathcal{A}_X \rightarrow \mathcal{B}_X$ is injective away from a subset of codimension 2;
2. \mathcal{C} is reflexive if and only if the localized map $\mu_X : \mathcal{A}_X \rightarrow \mathcal{B}_X$ is injective away from a subset of codimension 3.
3. \mathcal{C} is locally-free if and only if the localized map $\mu_X : \mathcal{A}_X \rightarrow \mathcal{B}_X$ is injective for all $X \in V$.

Proof. Dualizing the sequence (80) we obtain:

$$0 \rightarrow \mathcal{C}^* \rightarrow \mathcal{B}^* \xrightarrow{\mu^*} \mathcal{A}^* \rightarrow \text{Ext}^1(\mathcal{C}, \mathcal{O}_{\mathbb{P}^3}) \rightarrow 0 \quad (81)$$

It follows that $\text{Ext}^p(\mathcal{C}, \mathcal{O}_{\mathbb{P}^3}) = 0$ for $p = 2, 3$, and

$$I = \text{supp}(\text{Ext}^1(\mathcal{C}, \mathcal{O}_{\mathbb{P}^3})) = \{X \in V \mid \mu_X \text{ is not injective}\}$$

So it is now enough to argue that \mathcal{C} is torsion-free if and only if $\dim I = 1$ and that \mathcal{C} is reflexive if and only if $\dim I = 0$. The third statement is clear.

Recall that the m^{th} -singularity set of a coherent sheaf \mathcal{F} is given by:

$$S_m(\mathcal{F}) = \{X \in \mathbb{P}^3 \mid dh(\mathcal{F}_x) \geq 3 - m\}$$

where $dh(\mathcal{F}_x)$ stands for the homological dimension of \mathcal{F}_x as an \mathcal{O}_x -module:

$$dh(\mathcal{F}_x) = d \iff \begin{cases} \text{Ext}_{\mathcal{O}_x}^d(\mathcal{F}_x, \mathcal{O}_x) \neq 0 \\ \text{Ext}_{\mathcal{O}_x}^p(\mathcal{F}_x, \mathcal{O}_x) = 0 \quad \forall p > d \end{cases}$$

In the case at hand, we have that $dh(\mathcal{F}_x) = 1$ if $X \in I$, and $dh(\mathcal{F}_x) = 0$ if $X \notin I$. Therefore $S_0(\mathcal{C}) = S_1(\mathcal{C}) = \emptyset$, while $S_2(\mathcal{C}) = I$. It follows that [14, Proposition 1.20] :

- if $\dim I = 1$, then $\dim S_m(\mathcal{C}) \leq m - 1$ for all $m < 3$, hence \mathcal{C} is a locally 1st-syzygy sheaf;
- if $\dim I = 0$, then $\dim S_m(\mathcal{C}) \leq m - 2$ for all $m < 3$, hence \mathcal{C} is a locally 2nd-syzygy sheaf.

The desired statements follow from the observation that \mathcal{C} is torsion-free if and only if it is a locally 1st-syzygy sheaf, while \mathcal{C} is reflexive if and only if it is a locally 2nd-syzygy sheaf [12, page 148-149]. \square

C Quantum space-time calculations

We collect here the proofs of various facts used in Section 3.3.

First, we will derive a few formulas for the differential forms on quantum Minkowski space-time \mathfrak{M}_q^1 . The commutation relations between $x_{ij'}$ and $dx_{kl'}$ imply:

$$\begin{aligned} dx_{11'} \det(x) &= p^2 \det(x) dx_{11'} \\ dx_{12'} \det(x) &= p^2 q^{-2} \det(x) dx_{12'} \\ dx_{21'} \det(x) &= p^2 q^2 \det(x) dx_{21'} \\ dx_{22'} \det(x) &= p^2 \det(x) dx_{22'} \end{aligned} \quad (82)$$

We also obtain:

$$\begin{aligned} d(\det(x)) &= dx_{11'} x_{22'} + x_{11'} dx_{22'} - dx_{12'} x_{21'} - x_{12'} dx_{21'} = \\ &= p^{-1} q (x_{11'} dx_{22'} - x_{12'} dx_{21'}) + p^{-1} q^{-1} (x_{22'} dx_{11'} - x_{21'} dx_{12'}) \end{aligned} \quad (83)$$

Applying Leibnitz rule we obtain:

$$d(f \det(x)) = \left(\sum_{kl'} (\partial_{kl'} f) dx_{kl'} \right) \det(x) + f d(\det(x))$$

combining this with (82) and (83) we have:

$$\begin{aligned} \partial_{11'}(f \det(x)) &= p^2 (\partial_{11'} f) \det(x) + p^{-1} q^{-1} f x_{22'} \\ \partial_{12'}(f \det(x)) &= p^2 q^{-2} (\partial_{12'} f) \det(x) - p^{-1} q^{-1} f x_{21'} \\ \partial_{21'}(f \det(x)) &= p^2 q^2 (\partial_{21'} f) \det(x) - p^{-1} q f x_{12'} \\ \partial_{22'}(f \det(x)) &= p^2 (\partial_{22'} f) \det(x) + p^{-1} q f x_{11'} \end{aligned} \quad (84)$$

We introduce an operator:

$$\Delta_x f = (\partial_{11'} f) x_{11'} + (\partial_{12'} f) x_{12'} + (\partial_{21'} f) x_{21'} + (\partial_{22'} f) x_{22'} \quad (85)$$

and let D_x denote the operator of multiplication by $\det(x)$ on the right. Then we have:

Proposition 41.

$$\square_x D_x = p^4 D_x \square_x + p^2 \Delta_x + (p^{-2} + 1) \quad (86)$$

$$\Delta_x D_x = p^2 D_x \Delta_x + (p^{-2} + 1) D_x \quad (87)$$

$$D_x X_{nm}^l = p^{2l-1} [2l] X_{nm}^l \quad (88)$$

Proof. Using (84), we obtain:

$$\begin{aligned} \partial_{11'} \partial_{22'} (f \det(x)) &= p^4 (\partial_{11'} \partial_{22'} f) \det(x) + p^{-1} q f + p q^{-1} (\partial_{22'} f) x_{22'} + \\ &\quad + p q (\partial_{11'} f) x_{11'} + (p q - 1) (\partial_{12'} f) x_{12'} + \\ &\quad (p q - 1) (\partial_{21'} f) x_{21'} + p^{-1} q^{-1} (p q - 1)^2 (\partial_{22'} f) x_{22'} , \end{aligned}$$

and

$$\begin{aligned} \partial_{21'} \partial_{12'} (f \det(x)) &= p^4 (\partial_{21'} \partial_{12'} f) \det(x) - p^{-1} q^{-1} f - p q^{-1} (\partial_{12'} f) x_{12'} - \\ &\quad - p q^{-1} (\partial_{21'} f) x_{21'} - (p q^{-1} - 1) (\partial_{11'} f) x_{11'} - (p q^{-1} - q^{-2}) (\partial_{22'} f) x_{22'} . \end{aligned}$$

This yields (86). Again using (84), we have:

$$\begin{aligned} \sum_{kl'} (\partial_{kl'} (f \det(x))) x_{kl'} &= (p^2 (\partial_{11'} f) \det(x) + p^{-1} q^{-1} f x_{22'}) x_{11'} + \\ &\quad + (p^2 q^{-2} (\partial_{12'} f) \det(x) - p^{-1} q^{-1} f x_{21'}) x_{12'} + \\ &\quad + (p^2 q^2 (\partial_{21'} f) \det(x) - p^{-1} q f x_{12'}) x_{21'} + \\ &\quad + (p^2 (\partial_{22'} f) \det(x) + p^{-1} q f x_{11'}) x_{22'} , \end{aligned}$$

which yields (87).

For the third identity, we have already computed dX_{nm}^l in the proof of Proposition 28. The expression for $D_x X_{nm}^l$ is obtained from the one for dX_{nm}^l by replacing $dx_{kl'}$ by $x_{kl'}$. \square

The formulas in Proposition 41 easily imply that the elements (64) are the eigenfunctions of Δ_x and $\tilde{\square}_x$. Let:

$$\tilde{\square}_x (\det(x))^k X_{nm}^l = c_k (\det(x))^k X_{nm}^l$$

$$\Delta_x (\det(x))^k X_{nm}^l = d_k (\det(x))^k X_{nm}^l$$

Then the commutation relations (86) and (87) yield the following recurrent formulas:

$$c_{k+1} = p^4 c_k + p^2 d_k + p^{-2} + 1 \quad , \quad \text{and} \quad , \quad d_{k+1} = p^2 d_k + p^{-2} + 1$$

and we know from Propositions 28 and 41 that $c_0 = 0$ and $d_0 = p^{2l-1}[2l]$. A simple induction yields $c_k = p^{2k+2l-3}[k][k+2l+2]$, as claimed in Proposition 29.

Finally, we will prove Proposition 32. We note first a well-known fact in q -calculus: if

$$\begin{aligned} ab &= q^{-2}ba \\ \text{then} \quad (a+b)^n &= \sum_r \left\{ \begin{matrix} n \\ r \end{matrix} \right\} a^r b^{n-r} \\ \text{where} \quad \left\{ \begin{matrix} n \\ r \end{matrix} \right\} &= \frac{\{n\}!}{\{r\}!\{n-r\}!} \quad , \\ \{n\} &= \{1\} \dots \{n\} \quad , \quad \text{and} \quad \{n\} = \frac{q^{2n} - 1}{q^2 - 1} \quad . \end{aligned} \tag{89}$$

Now the expression (57) immediately implies that, up to a constant, X_{nm}^l is equal to a sum:

$$\sum_r \frac{x_{11'}^r}{\{r\}!} \frac{x_{21'}^{l-m-r}}{\{l-m-r\}!} \frac{x_{12'}^{l-n-r}}{\{l-n-r\}!} \frac{x_{22'}^{r+m+n}}{\{r+m+n\}!} \tag{90}$$

Since $\det(x) = (\det(y))^{-1}$, it is enough to show that (90) is equal to $(\det(x))^{2l} Y_{nm}^l$. Expanding as above we get:

$$(\det(x))^{2l} \sum_r \frac{y_{11'}^r}{\{r\}!} \frac{y_{12'}^{l-n-r}}{\{l-n-r\}!} \frac{y_{21'}^{l-m-r}}{\{l-m-r\}!} \frac{y_{22'}^{r+m+n}}{\{r+m+n\}!} \tag{91}$$

Expressing $y_{kl'}$ as $x_{kl'}/\det(x)$, commuting all factors of $(\det(x))^{-1}$ to the left and finally commuting $x_{12'}^{l-n-r}$ and $x_{21'}^{l-m-r}$, we reduce (91) to a sum of monomials as in (90), but with certain coefficients given by powers of q . An easy calculation shows that these powers do not depend on the summation index r , implying that the expressions in (90) and (91) are indeed proportional. This concludes the proof of Proposition 32.

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